Introduction to Formal Realizability Theory—II

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(Manuscript received February 15, 1952)

This part of the paper exhibits a network to realize a given positive real impedance matrix.

1. INTRODUCTION TO PART II

1.0 In this part of the paper we prove the following theorem:

1.1 *Theorem:* Let $Z(p)$ be an $n \times n$ matrix whose elements are $Z_{rs}(p)$, $1 \leq r, s \leq n$, where

(i) Each $Z_{rs}(p)$ is a rational function

(ii) $\overline{Z_{rs}(p)} = Z_{rs}(\bar{p})$

(iii) $Z_{rs}(p) = Z_{sr}(p)$

(iv) For each set of real constants $k_1, \cdots, k_n$, the function

$$\varphi_Z(p) = \sum_{r,s=1}^{n} Z_{rs}(p) k_r k_s$$

has a non-negative real part whenever $Re(p) > 0$.

Then there exists a finite passive network, a $2n$-pole, which has the impedance matrix $Z(p)$. A dual result holds for admittance matrices $Y(p)$.

1.2 The converse of this theorem was proved in Part I: that if a finite passive $2n$-pole has an impedance matrix $Z(p)$, then this matrix has properties (i) through (iv) of 1.1.

1.3 We recall that in Part I matrices satisfying the conditions of 1.1 were called *positive real* (PR).

1.4 The proof of 1.1 is a direct generalization to matrices of the Brune process[2] for realizing a two-pole impedance function $f(p)$. For this proof we shall require from Part I certain specific properties of positive real operators and matrices. These will be summarized in Section 2 below. Further than this, the present part is almost independent of Part I,

although in terminology, notation, and method a direct continuation of it. References to sections or paragraphs in Part I will be made thus: (I, 6) or (I, 6.23).

1.5 The distinction emphasized in Part I between operators, as abstract geometrical objects, and matrices as concrete arrays of numbers representing these geometrical objects, is not one which we have now to maintain with any strictness. We shall generally preserve it verbally but not use the bracket notation for matrices introduced in Part I.

II. PROPERTIES OF POSITIVE REAL OPERATORS AND MATRICES

2.0 We recall that an impedance operator $Z(p)$ is a linear function from the linear space **K** of current vectors k to the linear space **V** of voltage vectors v. A positive real operator $Z(p)$ is one whose matrix in any real coordinate frame is positive real. In Section 16 of Part I the following properties of a PR operator $Z(p)$ were established:

2.01 $Z(p)$ has no poles in Γ_+.*

2.02 If $Re(Z(p)k, k) = 0$ for some $p \in \Gamma_+$, then $Z(p)k \equiv 0$ for all p.

2.03 If it exists, $Z^{-1}(p) = Y(p)$ is PR.

2.04 If $Z(p)$ has a pole at $p = p_0$, it has one at $p = \bar{p}_0$.

2.05 If $Z(p)$ has a pole at $p = i\omega_0$, that pole is simple and

$$Z(p) = \frac{2p}{p^2 + \omega_0^2} R + Z_1(p),$$

where R is real, symmetric, semidefinite, and not zero, and $Z_1(p)$ is PR.

2.06 If $Z(p)$ has a pole at $p = \infty$, that pole is simple and

$$Z(p) = pR + Z_1(p)$$

where R and $Z_1(p)$ are as in 2.05.

2.07 It was emphasized at several points in Part I that the fact of possessing an impedance matrix, and that of possessing an admittance matrix, are each restrictions on a $2n$-pole **N**. It is readily verified from (I, 6.3) and (I, 6.31)—and, indeed, well known—that if **N** has both an impedance matrix $Z(p)$ and an admittance matrix $Y(p)$, then

$$Y(p) = Z^{-1}(p).$$

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* Γ_+ is the open right half plane: all finite p such that $Re(p) > 0$.

That is, if the impedance matrix of a $2n$-pole **N** is non-singular, then its admittance matrix exists, and conversely.

2.08 It was proved by Cauer[5], and in (I, 16.8), that if $Z(p)$ is PR and of rank $m < n$, then there exists a real, constant, non-singular matrix W such that

$$Z(p) = W'Z^B(p)W \tag{1}$$

where $Z^B(p)$ is a non-singular $m \times m$ PR matrix bordered by zeros.

2.09 Properties (i) through (iv) of 1.1 define the PR property for a matrix $Z(p)$. A convenient equivalent definition is that

(i) $Z(p)$ is symmetric,

(ii) For each $k \in \mathbf{K}$, the function

$$\varphi(p) = (Z(p)k, k)$$

is a PR function of p.

This equivalent definition was established in (I, 16.13).

2.1 In Section 16 of Part I it was also mentioned that there exists for any rational operator $Z(p)$ (PR or not) a numerical function $\delta(Z)$ which generalizes to operators the usual definition of the degree of a rational function. We list here the properties of this degree $\delta(Z)$. They will be established in Section 7.

2.11 $\delta(Z)$ is an integer ≥ 0.

2.12 If $\delta(Z) = 0$, then $Z(p)$ is a constant—that is, does not depend upon p.

2.13 If $Z^{-1}(p)$ exists, then $\delta(Z) = \delta(Z^{-1})$.

2.14 If $Z(p) = Z_1(p) + Z_2(p)$, where $Z_1(p)$ is finite at every pole of $Z_2(p)$, and $Z_2(p)$ is finite at every pole of $Z_1(p)$, then

$$\delta(Z) = \delta(Z_1) + \delta(Z_2).$$

2.15 If $Z(p) = f(p)R$, where $f(p)$ is a scalar and R is a constant operator, then

$$\delta(Z) = [\text{degree of } f]\cdot[\text{rank of } R].$$

Here the degree of f is the sum

$$\sum_{p_0} [\text{order of the pole of } f(p) \text{ at } p_0]$$

where p_0 runs over all poles of $f(p)$, including ∞.

2.16 If A and B are constant non-singular matrices, then

$$\delta(Z) = \delta(AZB).$$

It is evident then that $\delta(Z)$ is a geometrical property, being constant over tbe usual equivalence classes

$$W'Z(p)W$$

or

$$W^{-1}Z(p)W$$

of matrices. Hence we may speak of the degree $\delta(Z)$ of an operator $Z(p)$.

2.17 If $Z(p)$ is formed from an $m \times m$ matrix $Z_1(p)$ by bordering the latter with zeros, then

$$\delta(Z) = \delta(Z_1).$$

2.18 Concerning the degree $\delta(Z)$ we here state a fundamental tbeorem:

Theorem: The $2n$-pole whose existence is asserted by 1.1 can be constructed with $\delta(Z)$ reactive elements, and no fewer.

The proof of this theorem will be distributed through Sections 4 and 6. In fact, we must even define exactly the phrase "can be constructed with x reactive elements." This will be done in Section 3.

2.2 *Lemma:* If $Z_1(p)$ and $Z_2(p)$ are PR operators or matrices, then

$$Z(p) = Z_1(p) + Z_2(p)$$

is also PR. If either of $Z_1(p)$ or $Z_2(p)$ is non-singular, then $Z(p)$ is.

Proof: Clearly $Z(p)$ is symmetric. By 2.09, therefore, $Z(p)$ is PR if the function

$$(Z(p)k, k) = (Z_1(p)k, k) + (Z_2(p)k, k) \tag{1}$$

is PR for each $k \in \mathbf{K}$. The right hand side is obviously PR by hypothesis.

If either of $Z_i(p)$ is non-singular, the function (1) cannot vanish in Γ_+ unless $k = 0$ (this is 2.02). Hence in this case $Z(p)$ also is non-singular.

2.21 Clearly 2.2 is independent of the implication, tacit in the notation, tbat the operators involved are impedances. The lemma holds for PR operators, whether interpreted as operating from $\mathbf{K}$ to $\mathbf{V}$ (impedances) or from $\mathbf{V}$ to $\mathbf{K}$ (admittances).

2.3 In (I, 6.21) and (I, 6.3) it was noted that any $n \times n$ impedance matrix $Z(p)$ defines by fiat a general $2n$-pole $\mathbf{N}$ whose impedance matrix is that $Z(p)$. Such is the generality of the notion of general $2n$-pole (I, 4).

Given $2n$-poles $\mathbf{N}_1$ and $\mathbf{N}_2$, with impedance matrices respectively $Z_1(p)$ and $Z_2(p)$, we know then that there is a general $2n$-pole $\mathbf{N}$ whose impedance matrix is

$$Z(p) = Z_1(p) + Z_2(p).$$

We call this $\mathbf{N}$ the series combination of $\mathbf{N}_1$ and $\mathbf{N}_2$.

2.31 Designate the terminal pairs of $\mathbf{N}_1$ by (S_r, S_r'), those of $\mathbf{N}_2$ by (T_r, T_r'), $1 \leq r \leq n$. It is evident that if $\mathbf{N}_1$ and $\mathbf{N}_2$ appear in a diagram so connected that

(i) S_r' is connected to T_r, $1 \leq r \leq n$;

(ii) No other connections are made to these nodes;

then the device with terminals S_r, T_r' is $\mathbf{N}$. This follows at once from Kirchoff's laws applied to the ideal graph (I, 4.11).

2.32 Dually, if $\mathbf{N}_1$ and $\mathbf{N}_2$ have admittance matrices $Y_1(p)$, $Y_2(p)$, then

$$Y(p) = Y_1(p) + Y_2(p)$$

is the matrix of a $2n$-pole $\mathbf{N}$ defined as the parallel connection of $\mathbf{N}_1$ and $\mathbf{N}_2$. $\mathbf{N}$ is the device whose terminal pairs are formed by joining S_r, T_r and also S_r', T_r', $1 \leq r \leq n$.

2.33 Fig. 1 shows the conventions to be used in indicating $2n$-poles ($n = 4$ in the Figure) with, respectively, impedance matrices and admittance matrices. Fig. 2 then shows the series connection of two impedance devices and the parallel connection of two admittance devices. In each case the terminals on the left are those of the composite device.

2.4 The series and parallel connections just described are special ways of combining $2n$-poles needed for the generalized Brune process for matrices. They have been introduced here on their merits, as new op-

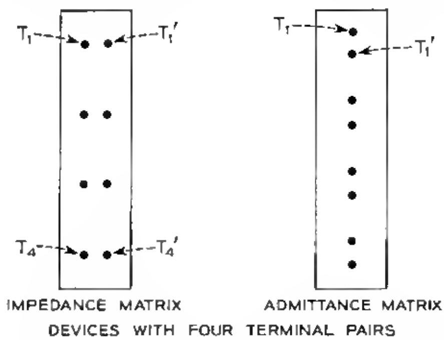


Fig. 1—Conventions used in representing 2N poles.

erations. They are, however, expressible in terms of the basic operations of juxtaposition (I, 17) and restriction (I, 18).

For example, the series connection of $\mathbf{N}_1$ and $\mathbf{N}_2$ is formed by first juxtaposing $\mathbf{N}_1$ and $\mathbf{N}_2$, to get a $2 \times 2n$-pole $\bar{\mathbf{N}}$. Let $\mathbf{J}$ be the $2n$ dimensional space of $2n$-tuples

$$j = [j_1, \cdots, j_n, \ell_1, \cdots, \ell_n].$$

We interpret this j as a $2n$-tuple of currents in the $2 \times 2n$-pole $\bar{\mathbf{N}}$, where j_r is the current in the r^{th} terminal pair of $\mathbf{N}_1$ and ℓ_r that in the r^{th} pair of $\mathbf{N}_2$, $1 \leq r \leq n$. Let $\mathbf{K}$ be an n-dimensional space. Given an n-tuple $k \in \mathbf{K}$, say

$$k = [k_1, \cdots, k_n],$$

we define the operator C from $\mathbf{K}$ to $\mathbf{J}$ by

$$j = Ck = [k_1, \cdots, k_n, k_1, \cdots, k_n].$$

Restricting $\bar{\mathbf{N}}$ by C gives the series combination $\mathbf{N}$ of $\mathbf{N}_1$ and $\mathbf{N}_2$. The details may easily be supplied by the interested reader.

2.41 Representing the series and parallel connections in terms of juxtaposition and restriction makes the lemma, 2.2, an immediate consequence of the lemma of (I, 17.2) and the theorems of (I, 17.3, 18.3).

2.5 We report here for record a curious property of PR operators which has so far found no application:

Lemma: If $Z(p)$ is a PR impedance operator from $\mathbf{K}$ to $\mathbf{V} = \mathbf{K}^*$, and $Y(p)$ any PR admittance operator from $\mathbf{V}$ to $\mathbf{K}$, then the operator

$$1 + Y(p)Z(p)$$

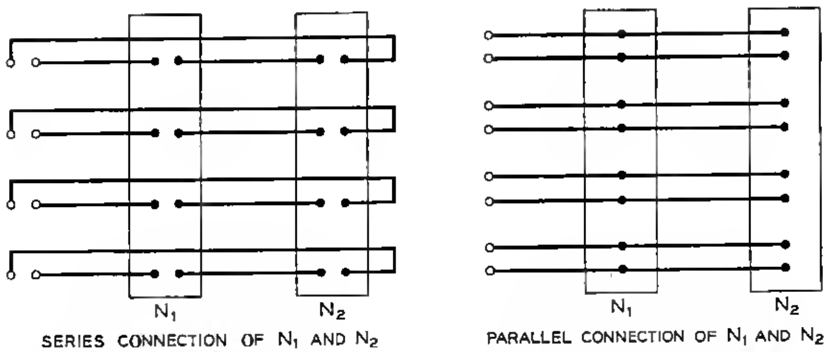


Fig. 2—Series and parallel connection of 2N poles: Series, left, and parallel, right.

in **K** is non-singular. Dually

$$1 + Z(p)Y(p)$$

in **V** is non-singular.

Proof: Suppose that $k \in \mathbf{K}$ is such that

$$(1 + Y(p)Z(p))k = 0 \tag{1}$$

for all p. Then

$$0 = Z(\bar{p})(1 + Y(p)Z(p))k = Z(\bar{p})k + Z(\bar{p})Y(p)Z(p)k$$

for all p. Then, however,

$$(Z(\bar{p})k, k) + (Z(\bar{p})Y(p)Z(p)k, k) = 0. \tag{2}$$

We may write the second term as

$$\overline{(Z^*(\bar{p})k, \overline{Y(p)Z(p)k})} = (Z(p)k, Y^*(p)Z^*(\bar{p})k) \tag{3}$$

by (I, 14.0) applied twice. Now $Z(p)$ is PR, in particular real and symmetric, so

$$Z^*(\bar{p}) = \overline{Z^*(p)} = Z'(p) = Z(p).$$

Using a similar calculation with $Y(p)$, the quantity (3) becomes

$$(Z(p)k, Y(\bar{p})Z(p)k). \tag{4}$$

For each $p \in \Gamma_+$, we have $\bar{p} \in \Gamma_+$ and the first term of (2) has a non-negative real part. But for $\bar{p} \in \Gamma_+$, (4) is the conjugate of

$$\overline{(v, Y(\bar{p})v)} \tag{5}$$

where $v = Z(p)k$. Now (5) is a PR function of $\bar{p}$, hence has a non-negative real part for $\bar{p} \in \Gamma_+$, for any v. In particular therefore this is true for the v which, at p, makes (5) the conjugate of (4). Therefore (4) has a non-negative real part throughout Γ_+. It follows from (2) then that

$$Re(Z(\bar{p})k, k) = 0$$

for all $\bar{p} \in \Gamma_+$. By 2.02, then,

$$Z(p)k \equiv 0.$$

By (1), then

$$1k = k = 0.$$

Hence (1) implies $k = 0$. Therefore the operator in (1) has an inverse

III. A SIMPLE REALIZABILITY THEOREM

3.0 The following theorem is contained in Cauer[5]. Since it provides the basic step in our realizability process, we shall prove it here.

3.1 *Theorem:* Let $f(p)$ be any one of the four functions

(i) $f(p) \equiv 1,$

(ii) $f(p) \equiv p,$

(iii) $f(p) = \dfrac{1}{p}$

(iv) $f(p) = \dfrac{2p}{p^2 + \omega_0^2}, \qquad \omega_0^2 > 0.$

Let R be a real, constant, symmetric semidefinite $n \times n$ matrix of rank r. Then:

(A) The matrix

$$Z(p) = f(p)R$$

is PR and there exists a finite passive $2n$-pole **N** with the impedance matrix $Z(p)$.

(B) The $2n$-pole **N** can be realized with ideal transformers and, respectively,

(i) with r resistors,
(ii) with r coils,
(iii) with r capacitors,
(iv) with r coils and r capacitors.

(C) The dual statements to (A) and (B) are true.

Proof: That $Z(p)$ in PR is easily verified directly. It will follow also from the results of Part I when we exhibit a (finite passive) network whose matrix is $Z(p)$. To construct this latter, let D be a diagonal matrix such that

$$R = WDW'$$

where W is a real, constant, non-singular matrix. That D and W always exist is the analog for impedance operators of the result of Halmos[9], par. 41, for dimensionless operators. In fact, W can be taken to be orthogonal ($W^{-1} = W'$, cf. Halmos[9], par. 63). If R is of rank r, D has r non-vanishing diagonal elements, say $d_{11}, d_{22}, \cdots, d_{rr}$.

Since R is semidefinite, each $d_{ii}f(p)$, $1 \leq i \leq r$, is the impedance of an obviously passive two pole. Call this two-pole $\mathbf{M}_i$. Let $\mathbf{M}_{r+1}, \cdots, \mathbf{M}_n$ be two poles consisting of short circuits. Consider the $2n$-pole $\mathbf{N}_1$

made by connecting $\mathbf{M}_s$ between T_s and T'_s, $1 \leq s \leq n$. This $2n$-pole has the impedance matrix

$$Z_1(p) = f(p)D.$$

Then

$$Z(p) = f(p)WDW' = WZ_1(p)W'$$

is the matrix of a $2n$-pole $\mathbf{N}$ which can be obtained from $\mathbf{N}_1$ by the use of ideal transformers. Clearly $\mathbf{N}_1$, and therefore $\mathbf{N}$, contains exactly the elements claimed in (B) of the theorem.

The dual theorem (C) needs no comment.

3.11 *Corollary:* The conclusion (A) of 3.1 holds if the hypotheses on $f(p)$ are replaced by "$f(p)$ is PR." The same method of proof applies but one must use the Brune theory to realize the impedances $d_{ii}f(p)$, $1 \leq i \leq r$.

3.2 The case (ii) of 3.1 shows that any physical system of coupled coils can be realized with a set of isolated (i.e., not coupled) coils, with ideal transformers to supply the coupling [Cf. (I, 19.12)]. With this fact in mind, we see that the method of network synthesis used in (I, 19) can be simplified to the following: one starts with a finite collection of two-poles: each one is a resistor, capacitor, or coil (inductor). These are then appropriately connected to suitable ideal transformers. Viewed from certain selected terminals of these transformers, this network is a $2n$-pole equivalent to the desired one.

The difference between this process and that of (I, 19) is the minor one that coupled coils have been eliminated. We may then, however, regard any finite passive network as made up solely of simple two-poles (resistors, capacitors, coils) and ideal transformers.

It is readily verified from (I, 19.2) that open and short circuits are special cases of ideal transformers.

If a network made up in this way uses ℓ coils and c capacitors, we shall call $\ell + c$ the number of reactive elements in the network (or used by, or used in, the network).*

3.21 *Lemma:* The network described in the proof of 3.1 uses $\delta(Z)$ reactive elements. This is obvious from 2.12, 2.15, and 2.16.

IV. THE BRUNE PROCESS FOR A POSITIVE REAL MATRIX

4.0 Let $Z(p)$ be an $n \times n$ PR matrix. We can regard it as the impedance matrix of a general $2n$-pole $\mathbf{N}$. In this section we shall describe the

* By this definition, a reactive element is an energy storage element. Ideal transformers are not reactive, by the very fact of their ideality.

construction of a finite passive network which, as a $2n$-pole, has the impedance matrix $Z(p)$—i.e. is a $2n$-pole equivalent to $\mathbf{N}$. We call such a network a (physical) realization of $\mathbf{N}$, or of $Z(p)$. The dual problem, that of realizing a PR admittance matrix, can be handled dually.

Let $Z_0(p) = Z(p)$, $\mathbf{N}_0 = \mathbf{N}$, $n_0 = n$. We describe an inductive procedure which, given a $2n_r$-pole $\mathbf{N}_r$, $r \geq 0$, either

(i) Constructs a physical realization of $\mathbf{N}_r$, or

(ii) Constructs a $2n_{r+1}$-pole $\mathbf{N}_{r+1}$ such that if $\mathbf{N}_{r+1}$ is physically realizable, then $\mathbf{N}_r$ is.

To show that this induction actually gives a realization of any PR matrix $Z_0(p)$ we must demonstrate that, first, it is effective—i.e. that at any stage $\mathbf{N}_r$ at least one of (i) and (ii) is possible. Second, we must show that the process terminates with the construction of a finite network. The details of these demonstrations are given in the paragraphs 4.1 et seq. of this section. In the paragraphs 4.01 to 4.07 we describe the logical pattern into which these details are to be fit when they are established.

4.01 There are nine basic operations by which the networks $\mathbf{N}_r$ are constructed. We name the operations here, in order to give a clearer picture of the logic of the process, but their mathematical treatment is deferred to later paragraphs.

IP: A PR impedance matrix $Z_r(p)$ which has poles on $p = i\omega$ is represented as

$$Z_r(p) = pR_\infty + \frac{1}{p} R_0 + \sum_k \frac{2p}{p^2 + \omega_k^2} R_k + Z_{r+1}(p),$$

where $Z_{r+1}(p)$ is PR and has no poles on $p = i\omega$.

AP: A PR admittance matrix $Y_r(p)$ is represented dually:

$$Y_r(p) = pG_\infty + \frac{1}{p} G_0 + \sum_k \frac{2p}{p^2 + \omega_k^2} G_k + Y_{r+1}(p).$$

ID: A PR impedance matrix $Z_r(p)$ is represented as $W'Z_{r+1}^B(p)W$, where $Z_{r+1}^B(p)$ is a non-singular $Z_{r+1}(p)$ bordered by zeros.

AD: Dual to ID.

Res: A PR matrix $Z_r(p)$ is represented as

$$Z_r(p) = aS + Z_{r+1}(p),$$

where S is real, constant, symmetric, and positive definite, and $a \geq 0$ is the largest a for which $Z_{r+1}(p)$ is PR.

Con: The dual to Res.

IB: This is the analog of the step in the Brune process for scalars in which the reactance of a minimum resistance structure is tuned out to create a zero. The details are intricate in the generalization to $2n$-poles.

AB: This is the dual operation to IB.

F: A $2n_r$-pole $\mathbf{N}_r$ which has a constant PR matrix (admittance or impedance) is realizable at once, by 3.1. The operation **F** denotes this realization.

To each $\mathbf{N}_r$, one of these nine operations is to be applied. The effect of the last (F) is clearly salutary. That of each of the others is to split off a realizable piece of $\mathbf{N}_r$ and leave a $2n_{r+1}$-pole $\mathbf{N}_{r+1}$ to which again some one of the operations is to be applicable.

Exactly which of these operations to apply at any stage depends upon the properties of the $\mathbf{N}_r$ in question. We shall first devise a notation for describing the relevant properties of $\mathbf{N}_r$, and then in 4.04 present a table which summarizes what is to be proved in the paragraphs 4.1 et seq.

4.02 Definition: We say that $Z(p)$ has a zero of its real part at $p = i\omega_0$ if for some $k \in \mathbf{K}$, $k \neq 0$, we have

$$[Z(i\omega_0) + Z(-i\omega_0)]k = 0.$$

4.03 Let I be an integer describing a $2n$-pole $\mathbf{N}$ as follows:

$I = 0$ if $\mathbf{N}$ has no impedance matrix.

$I = 1$ if $\mathbf{N}$ has a non-constant impedance matrix which has no poles on $p = i\omega$, and no zeros of its real part on $p = i\omega$.

$I = 2$ if $\mathbf{N}$ has a non-constant impedance matrix with a zero of its real part on $p = i\omega$, but no poles on $p = i\omega$.

$I = 3$ if $\mathbf{N}$ has an impedance matrix with a pole or poles on $p = i\omega$.

Let A be an integer describing the admittance category of $\mathbf{N}$ in a dual way (e.g., $A = 0$ if $\mathbf{N}$ has no admittance matrix, etc.).

Let (I, A) denote the category of $2n$-poles $\mathbf{N}$ for which the indicated values of both I and A are true. Let

$$(I_1 + I_2, \qquad A_1 + A_2) \tag{1}$$

denote the category of $2n$-poles $\mathbf{N}$ for which either I_1 or I_2 is true and, simultaneously, either A_1 or A_2 is true, with a similar definition for more summands. Then for example the category (1) above consists of the logical union of the following:

$$(I_1, A_1), \qquad (I_1, A_2), \qquad (I_2, A_1), \qquad (I_2, A_2).$$

Let C denote the category of $2n$-poles $\mathbf{N}$ which have a constant matrix, impedance or admittance.

It is clear that any $2n$-pole **N** belongs in C or in exactly one of the sixteen elementary categories whose union is $(0 + 1 + 2 + 3, 0 + 1 + 2 + 3)$.

Table 4.04 shows for each category of $\mathbf{N}_r$, except (0, 0), which operations may be applied, and the possible categories of the resulting $\mathbf{N}_{r+1}$.

A $2n$-pole **N** not in (0, 0) has at least one matrix, and if it bas two these are of the same degree (2.07, 2.13). We may then denote the degree of whatever matrix **N** has simply by $\delta(\mathbf{N})$. The fourth and fifth columns of Table 4.04 show the relations of $\delta(\mathbf{N}_r)$ to $\delta(\mathbf{N}_{r+1})$, and of n_r to n_{r+1}.

4.05 Table 4.04 summarizes facts to be proved in 4.1 et seq. Assuming now that the assertions in this table are true, we can show that the inductive procedure is effective.

We observe first that the category C and every possible elementary category (I, A) except (0, 0) is contained in at least one of the categories listed in the first column. Hence to any $2n$-pole not in (0, 0) there is at least one operation applicable. Further we note that the category (0, 0) does not appear in the third column. Since by hypothesis $\mathbf{N}_0$ is not in (0, 0), it follows by induction that no $\mathbf{N}_r$ will be. Therefore the process can stop only by the operation F: completion.

Second, we notice that if $\mathbf{N}_r$ is not in the category (1, 1), then an applicable operation can be found which actually reduces one of the two numbers $\delta(\mathbf{N}_r)$, n_r. Furthermore, if $\mathbf{N}_r$ is in (1, 1), a sequence of two operations can be found which reduces one of $\delta(\mathbf{N}_r)$, n_r. Therefore before the realization process terminates (with F),

(i) There are not more operations chosen from the list IP, AP, IB, AB, than the integer $\delta(\mathbf{N}_0)$;

(ii) There are not more operations chosen from the list ID, AD, than the integer $n_0 - 1$ (since after these, still $n_{r+1} > 0$);

TABLE 4.04

Category of $\mathbf{N}_r$	Operation	Category of $\mathbf{N}_{r+1}$	$\delta(\mathbf{N}_r) - \delta(\mathbf{N}_{r+1})$	$n_r - n_{r+1}$
$(3, 0 + 1 + 2 + 3)$	IP	$C + (1 + 2, 0 + 1 + 2 + 3)$	>0	0
$(0 + 1 + 2 + 3, 3)$	AP	$C + (0 + 1 + 2 + 3, 1 + 2)$	>0	0
$(1 + 2, 0)$	ID	$(1 + 2, 1 + 2 + 3)$	0	>0*
$(0, 1 + 2)$	AD	$(1 + 2 + 3, 1 + 2)$	0	>0*
$(1, 1)$	Res	$(2, 0 + 1 + 2 + 3)$	0	0
$(1, 1)$	Con	$(0 + 1 + 2 + 3, 2)$	0	0
$(2, 1 + 2)$	IB	$C + (1 + 2 + 3, 0 + 1 + 2 + 3)$	>0	0
$(1 + 2, 2)$	AB	$C + (0 + 1 + 2 + 3, 1 + 2 + 3)$	>0	0
C	F	—	—	—

* But $n_{r+1} > 0$.

(iii) There are not more operations chosen from the list Res, Con, than the integer $\delta(\mathbf{N}_0) + n_0 - 1$.

Finally, then, the process must terminate after at most $2\delta(\mathbf{N}_0) + 2n_0 - 1$ operations.

4.06 Besides the data in 4.04, one other fact must be established about each operation: that $\mathbf{N}_r$ is physically realizable if $\mathbf{N}_{r+1}$ is. This will be done as we discuss each operation. When it is established, we reason back from the result of operation F, which provides a physical realization of some $\mathbf{N}_m$ ($m \leq 2\delta(\mathbf{N}_0) + n_0 - 1$), through $\mathbf{N}_{m-1}$ to $\mathbf{N}_0 = \mathbf{N}$, and obtain a realization of $\mathbf{N}$ in finitely many steps.

4.07 Finally, we shall prove about each step that:

If $\mathbf{N}_{r+1}$ can be realized with x_{r+1} reactive elements, then $\mathbf{N}_r$ can be realized with

$$x_{r+1} + \delta(\mathbf{N}_r) - \delta(\mathbf{N}_{r+1})$$

reactive elements. This observation will provide the basis for proving the theorem of 2.18. For if $\mathbf{N}_m$ is the network with which the process terminates, then by 3.21 $\mathbf{N}_m$ can be realized with $\delta(\mathbf{N}_m)$ reactive elements. Reading back through the construction, each increment of degree that is encountered is balanced by an equal increment in the total number of reactive elements, so that, finally, $\delta(\mathbf{N})$ is tbe total number of reactive elements used. That no construction using fewer reactive elements can succeed will be shown in Section 6.

We now turn to IP, ID, Res, and IB, omitting the dual considerations. In each case, notation is simplified by writing Z, Y, $\mathbf{N}$, n respectively for Z_r, Y_r, $\mathbf{N}_r$, n_r, and Z_1, Y_1, $\mathbf{N}_1$, n_1 for Z_{r+1}, Y_{r+1}, $\mathbf{N}_{r+1}$, n_{r+1}.

4.1 Given a $2n$-pole $\mathbf{N}$ in any category for which $I = 3$, its impedance matrix $Z(p)$ exists by hypothesis and has poles on $p = i\omega$. These can be removed successively by 2.05 and 2.06, giving

$$Z(p) = pR_\infty + \frac{1}{p} R_0 + \sum_{k=1}^{K} \frac{2p}{p^2 + \omega_k^2} R_k + Z_1(p). \tag{1}$$

In this expansion, either of R_0, R_∞ may of course he absent, and all the R_k are real, symmetric, constant and semidefinite, for $k = 0, 1, \cdots, K, \infty$. Furthermore, $Z_1(p)$ is PR and has no poles on $p = i\omega$, by 2.05, 2.06 and construction.

Let $\mathbf{N}_1$ be the $2n_1$-pole whose impedance matrix is $Z_1(p)$. We define IP to be the operation giving $\mathbf{N}_1$ from $\mathbf{N}$. Either $\mathbf{N}_1 \in C$, or $I = 1$ or 2

for $\mathbf{N}_1$, since at least $Z_1(p)$ exists. Furthermore, by construction $Z_1(p)$ is again an $n \times n$ matrix, so $n_1 = n$.

By 2.14 and 2.15,

$$\delta(Z) = \text{rank }(R_\infty) + \text{rank }(R_0) + 2\sum_{k=1}^{K} \text{rank }(R_k) + \delta(Z_1). \tag{2}$$

Since $\delta(Z)$ is finite, this shows that K is finite. Indeed, $2K \leq \delta(Z)$. Furthermore, $\delta(Z) > \delta(Z_1)$, because a matrix of rank zero is itself zero, and by hypothesis $Z(p)$ has a pole on $p = i\omega$. Therefore we have established the claims in the first line of the Table 4.04, and by a dual argument those in the second line.

We must yet show that if $\mathbf{N}_1$ is physically realizable, then $\mathbf{N}$ is. Each term in (1), save $Z_1(p)$, is the matrix of a physically realizable $2n$-pole, by 3.1. There are at most $K + 2$ such terms. The series combination of their respective $2n$-poles is therefore physically realizable and $\mathbf{N}$ results from the series connection of these and $\mathbf{N}_1$ (2.2). Therefore if $\mathbf{N}_1$ is realizable, so is $\mathbf{N}$.

Fig. 3 shows the relation of $\mathbf{N}$ and $\mathbf{N}_1$ under IP, and the dual relation under AP. Here we have shown $n = 3$. The boxes labelled 0, ∞, $\cdots$, K are the devices corresponding to the poles at $0, \infty, \cdots, i\omega_K$, the terminals on the extreme left are those of $\mathbf{N}$, and $\mathbf{N}_1$ is on the right.

4.11 From (2), and (B) of 3.1, we see that the number of reactive elements used in the realization of the network between $\mathbf{N}_1$ and $\mathbf{N}$ is exactly

$$\delta(Z) - \delta(Z_1) = \delta(\mathbf{N}) - \delta(\mathbf{N}_1).$$

Clearly the dual result holds for AP. This verifies 4.07 for IP and AP.

4.2 Consider a $2n$-pole $\mathbf{N}$ in $(1 + 2, 0)$. In particular, then, the impedance matrix $Z(p)$ of $\mathbf{N}$ exists and is not constant, but $Z(p)$ has no inverse. Then 2.08 applies, and we have

$$Z(p) = W' Z_1^B(p) W, \tag{1}$$

where W is real, constant, and non-singular, and $Z_1^B(p)$ is a non-singular matrix $Z_1(p)$ bordered by zeros. Let $\mathbf{N}_1$ be the $2n_1$-pole whose impedance matrix is $Z_1(p)$. We define ID as the operation which gives $\mathbf{N}_1$ from $\mathbf{N}$. Now $n_1 < n$, because $Z(p)$ is singular and $Z_1(p)$ is not. Also, $Z_1(p)$ is not constant, because $Z(p)$ is not, and $\delta(Z_1) = \delta(Z)$, by 2.17. Therefore $n_1 \neq 0$, also $\mathbf{N}_1$ is not in C. Because $Z_1(p)^{-1}$ exists, $\mathbf{N}_1$ is in $A = 1, 2$ or 3. Because $Z(p)$ has no poles on $p = i\omega$, neither has $Z_1(p)$, so $\mathbf{N}_1 \in (1 +$

2, 1 + 2 + 3). This verifies the statements on the third line of the Table 4.04, and the fourth by duality.

That **N** is physically realizable if $\mathbf{N}_1$ is, is the gist of (I, 8.11) and (I, 8.4). We prove it here by noting that $Z_1^B(p)$ is the matrix of a $2n$-pole $\mathbf{N}_2$ which obtains by adjoining $n - n_1 > 0$ pairs of shorted terminals to $\mathbf{N}_1$. Then (1) shows that **N** obtains from $\mathbf{N}_2$ by the use of ideal transformers (I, 9.1).

Fig. 4 shows in schematic form the effects of the operation ID and AD. In each case, it is emphasized that $\mathbf{N}_1$ has a matrix dual to that of **N**. We have shown $n = 5$, $n_1 = 3$.

4.21 No reactive elements are used in this construction, so 4.07 is satisfied.

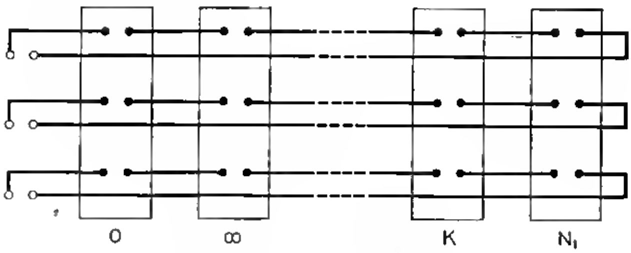
4.3 Consider now a $2n$-pole **N** in (1, 1). Then its impedance matrix $Z(p)$ is finite for every $p = i\omega$, and not constant.

Let $R(p)$, $I(p)$, respectively, be the real and imaginary parts of $Z(p)$:

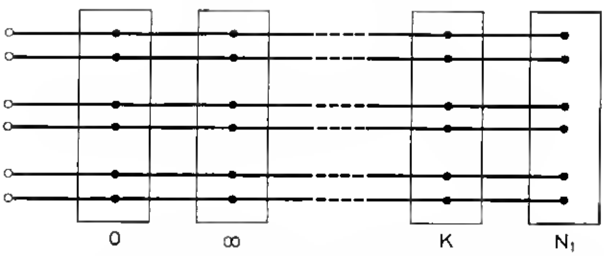
$$2R(p) = Z(p) + \overline{Z(p)} = Z(p) + Z(\bar{p}) = Z(p) + Z^*(p);$$

$$2iI(p) = Z(p) - \overline{Z(p)} = Z(p) - Z(\bar{p}) = Z(p) - Z^*(p);$$

$$Z(p) = R(p) + iI(p).$$



STRUCTURE RESULTING FROM IP



STRUCTURE RESULTING FROM AP

Fig. 3—Structure resulting from IP, above and AP, below.

Then $R(p) = R^*(p)$, $I(p) = I^*(p)$, and both are real and symmetric. If k is any vector,

$$(Z(p)k, k) = (R(p)k, k) + i(I(p)k, k),$$

and the self-adjoint property of R and I imply that each scalar product on the right is real. Therefore

$$\begin{aligned} Re(Z(p)k, k) &= (R(p)k, k), \\ Im(Z(p)k, k) &= (I(p)k, k). \end{aligned} \tag{1}$$

We note that

$$2iI(\bar{p}) = Z(\bar{p}) - Z^*(\bar{p}) = Z^*(p) - Z(p) = -2iI(p)$$

so that, in particular, $I(i\omega)$ is an odd function of ω.

4.31 *Lemma:* Let S be a given real, constant, symmetric, and positive *definite* matrix. Then there exists a unique number $a > 0$ such that

(i) The matrix

$$R(i\omega) - aS$$

is semidefinite for every ω,

(ii) For some $\omega_0 \geq 0$, possibly $+\infty$,

$$R(i\omega_0) - aS$$

is singular.

Proof: We first show how the number a would be calculated, and then reduce the claims of the lemma to a well-known and basic theorem in

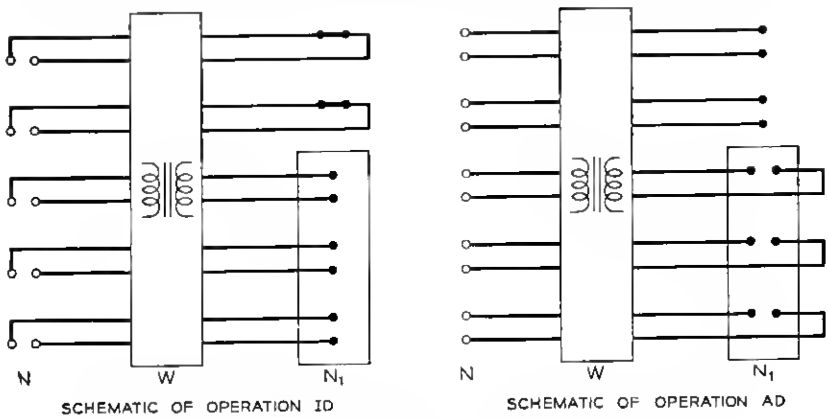


Fig. 4—Schematic of operation ID, left and AD, right.

the theory of quadratic forms. Fix ω and consider the matrix

$$R(i\omega) - \lambda S$$

as a function of λ. Its determinant,

$$\Delta_\omega(\lambda) = | R(i\omega) - \lambda S |,$$

is an n^{th} degree polynomial in λ with the following two properties:

(α) The coefficient of λ^n in $\Delta_\omega(\lambda)$ is not zero and is independent of ω,

(β) The n roots of

$$\Delta_\omega(\lambda) = 0 \tag{2}$$

are real and positive.

Now $R(i\omega)$ is rational, hence continuous, and finite for all ω, including $\omega = \infty$, by the hypothesis that **N** is in (1, 1). By (α) above, therefore, each root of (2) is a continuous function of ω on the compact set $-\infty \leq \omega \leq \infty$. Let $a(\omega)$ denote the least root of (2). Then $a(\omega)$ is again bounded and continuous for all ω. There is, therefore, an ω_0 where $a(\omega)$ takes its least value. This is the ω_0 referred to in the lemma, and

$$a \doteq a(\omega_0).$$

We see that this calculation requires solving an n^{th} degree polynomial equation containing a parameter (ω), and then minimizing the least root by varying the parameter. Though some properties of $R(i\omega)$ are available to assist in the process, and the choice of S is somewhat free to us, this is scarcely a feasible calculation in practice. Even when one reduces the minimizing problem to finding the roots of a derivative, there remains a prodigious calculation in all but the simplest cases.

Since by its definition $R(i\omega) = R(-i\omega)$, we may take $\omega_0 \geq 0$.

The relation (1) above implies that

$$(R(i\omega)k, k) \geq 0$$

for all real ω and all $k \in \mathbf{K}$, because $Z(p)$ is PR. That is, $R(i\omega)$ is semi-definite. The hypothesis that $Z(p)$ has no zero of its real part $R(p)$ on $p = i\omega$ then implies that $R(i\omega)$ is positive definite. All of (i), (ii), (α), and (β) then follow from well-known properties of definite quadratic forms. They may, for example, all be deduced from Halmos[9], paragraphs 62, 63, and 74, by choosing a coordinate frame in which the operator corresponding to S above is represented by the unit matrix. A more elegant reduction to the cited results of Halmos[9] can also be constructed.

4.32 *Lemma:* Given **N** in (1, 1), we choose any real constant symmetric

and positive definite matrix S and find the a described in 4.31. Then the matrix

$$Z_1(p) = Z(p) - aS$$

is PR and has a zero of its real part at $p = i\omega_0$.

Proof: Clearly $Z_1(p)$ is symmetric. By 2.09, then, $Z_1(p)$ is PR if the function

$$\varphi_1(p) = (Z_1(p)k, k) = (Z(p)k, k) - a(Sk, k) \tag{3}$$

is PR for each k. Clearly this function is rational and has no singularities in Γ_+. It suffices then to show that its real part is non-negative on $p = i\omega$. By (1) of 4.3

$$Re\, \varphi_1(i\omega) = (R(i\omega)k, k) - a(Sk, k)$$

and this is non-negative by (i) of 4.31.

That $Z_1(p)$ has a zero of its real part at $p = i\omega_0$ is (ii) of 4.31.

4.33 Let $\mathbf{N}_1$ be the $2n$-pole whose impedance matrix is the $Z_1(p)$ of 4.32. We define the operation Res as that which produces $\mathbf{N}_1$ from $\mathbf{N}$. It is evident from (3) above that the poles of $Z_1(p)$ are exactly those of $Z(p)$, hence $I = 2$ for $\mathbf{N}_1$. Nothing can be said of the admittance matrix for $\mathbf{N}_1$. $\delta(Z_1) = \delta(Z)$ by 2.14 and 2.15, and $n_1 = n$ by construction. The claims in 4.04 are now established for Res, and dually for Con.

The relation

$$Z(p) = Z_1(p) + aS$$

shows that $\mathbf{N}$ is a series combination of $\mathbf{N}_1$ and a device with the impedance matrix aS. Since $a > 0$, this latter is a realizable resistance network (3.1). Hence $\mathbf{N}$ is realizable if $\mathbf{N}_1$ is.

4.34 We observe that no reactive elements are used in the network between $\mathbf{N}_1$ and $\mathbf{N}$ (2.12, 3.12). This verifies 4.07 for Res and Con.

4.4 We now turn to the *piece de resistance* of the generalized Brune process, the operation IB and its dual. Consider a $2n$-pole $\mathbf{N}$ in the category $(2, 1 + 2)$—i.e., its impedance matrix $Z(p)$ exists, is not constant, is non-singular on $p = i\omega$, and has a zero of its real part at some $p = i\omega_0$. We have for some $k \in \mathbf{K}$ such that $k \neq 0$,

$$R(i\omega_0)k = 0. \tag{1}$$

Here, $R(p)$ is as defined in 4.3.

4.41 We now assert that we may assume that $0 < \omega_0$, and $i\omega_0 \neq \infty$ in

(1). Certainly we may take $\omega_0 \geq 0$, because $R(i\omega) = R(-i\omega)$. Furthermore, by (1),

$$Z(i\omega_0)k = iI(i\omega_0)k. \tag{2}$$

$I(i\omega)$, being odd, and finite everywhere on $p = i\omega$, must vanish at $\omega = 0$, and at $i\omega = \infty$. Hence if $\omega_0 = 0$ or $i\omega_0 = \infty$, $Z(i\omega_0)k = 0$ and $Z(p)$ is singular on $p = i\omega$. This denies our hypothesis that $\mathbf{N} \in (2, 1 + 2)$.

4.42 Let $\mathbf{J}$ be the set of all vectors $k \in \mathbf{K}$ such that (1) holds: the null space of $R(i\omega_0)$. Then clearly $\mathbf{J}$ is a linear manifold. Furthermore, $\mathbf{J}$ is real, because, if (1) holds then

$$\overline{R(i\omega_0)k} = \overline{R(i\omega_0)}\bar{k} = R(i\omega_0)\bar{k} = \bar{0} = 0$$

and $\bar{k}$ also is in $\mathbf{J}$.

Relations (1) and (2) hold for all $k \in \mathbf{J}$.

4.43 By its construction, $I(i\omega_0)$ is real and symmetric, but not necessarily definite. There does however exist a real diagonal matrix D and a real non-singular W such that $I(i\omega_0) = W'DW$. Let D_+ be the (diagonal) matrix obtained from D by replacing all negative elements of D by zero, and define D_- by

$$D = D_+ - D_- . \tag{3}$$

Then D_+ and D_- are real, symmetric, and non-negative semidefinite. Define

$$\begin{aligned} A &= \omega_0 W'D_+W, \\ B &= \frac{1}{\omega_0} W'D_-W. \end{aligned} \tag{4}$$

We have chosen $\omega_0 > 0$, so A and B are both real, symmetric and non-negative. Certainly therefore

$$Z^{(2)}(p) = Z(p) + \frac{1}{p} A + pB \tag{5}$$

is PR. Also $Z^{(2)}(p)$ has an inverse, because $Z(p)$ has one by hypothesis and 2.2 applies.

4.431 Let $v \in \mathbf{V}$ be such that for some $k_1 \in \mathbf{K}$

$$v = Ak_1$$

and for some $k_2 \in \mathbf{K}$

$$v = Bk_2 .$$

Then $v = 0$.

Proof: We may assume that the first r diagonal elements of D are the non-zero elements of D_+, the next s those of $-D_-$. By (4),

$$(W')^{-1}v = \omega_0 D_+ W k_1,$$

$$(W')^{-1}v = \frac{1}{\omega_0} D_- W k_2.$$

The first of these relations exhibits $(W')^{-1}v$ as an n-tuple with non-zero components at most among the first r, the second as an n-tuple with non-zero components at most among the last $n - r$. Hence all components of $(W')^{-1}v$ are zero. Hence v itself is zero.

4.44 Define

$$X(p) = -\frac{1}{p} A - pB, \tag{6}$$

and let $\mathbf{N}_X$ be the $2n$-pole whose impedance matrix is $X(p)$. $\mathbf{N}_X$ is not physically realizable, since it is made up of negative reactances.

Let $\mathbf{N}^{(2)}$ be the $2n$-pole whose impedance matrix is $Z^{(2)}(p)$. Then by (5) $\mathbf{N}$ obtains from $\mathbf{N}^{(2)}$ and $\mathbf{N}_X$ by connecting them in series.

We have the following relation holding on $p = i\omega$, but only thereon since it is only there that $X(p)$ is a pure imaginary:

$$Z^{(2)}(i\omega) = R(i\omega) + i\left[I(i\omega) - \frac{1}{\omega} A + \omega B\right].$$

In particular, at $i\omega_0$,

$$\begin{aligned} Z^{(2)}(i\omega_0) &= R(i\omega_0) + i[I(i\omega_0) - W'D_+W + W'D_-W] \\ &= R(i\omega_0), \end{aligned}$$

by (3) and (4). Since $\mathbf{J}$ is the null space of $R(i\omega_0)$ by definition, $\mathbf{J}$ is the null space of $Z^{(2)}(i\omega_0)$.

4.45 Now $Y^{(2)}(p) = [Z^{(2)}(p)]^{-1}$ exists and is PR. Since $Z^{(2)}(i\omega_0)$ annihilates every element of $\mathbf{J}$, it follows that $Y^{(2)}(p)$ does not exist at $p = i\omega_0$—therefore $Y^{(2)}(p)$ has a pole at $i\omega_0$. Hence we may apply AP and represent $Y^{(2)}(p)$ as a reactance network, with admittance matrix

$$G(p) = \frac{2p}{p^2 + \omega_0^2} G, \tag{7}$$

in parallel with a $2n$-pole $\mathbf{N}^{(3)}$ which has an admittance matrix, say

$$Y^{(2)}(p) = G(p) + Y^{(3)}(p), \tag{8}$$

where $Y^{(3)}(p)$ is finite at $p = i\omega_0$.

4.46 Multiplying (8) on either side by $Z^{(2)}(p)$,

$$\begin{aligned}\frac{2p}{p^2+\omega_0^2}GZ^{(2)}(p) + Y^{(3)}(p)Z^{(2)}(p) &= 1 \\ &= \frac{2p}{p^2+\omega_0^2}Z^{(2)}(p)G + Z^{(2)}(p)Y^{(3)}(p).\end{aligned} \tag{9}$$

Here, to be strictly correct, we should write two separate equations, interpreting 1 as the identity operator in **K** for, here, the left equality, and as the identity operator in **V** for the right equality. Multiplying (9) through by $p - i\omega_0$ and letting $p \rightarrow i\omega_0$, we obtain

$$GZ^{(2)}(i\omega_0) = 0 = Z^{(2)}(i\omega_0)G.$$

Here, as in (9), we have condensed two dimensionally incompatible equalities. From this it follows that each of G and $Z^{(2)}(i\omega_0)$ has its range in the null space of the other. In particular, therefore, the range of G is contained in **J**.

4.47 Consider now a v such that $Gv = 0$. Then, by (7) and (8),

$$v \equiv Z^{(2)}(p)Y^{(2)}(p)v \equiv Z^{(2)}(p)Y^{(3)}(p)v$$

so, at $i\omega_0$,

$$v = Z^{(2)}(i\omega_0)Y^{(3)}(i\omega_0)v = Z^{(2)}(i\omega_0)k$$

for some finite vector $k = Y^{(3)}(i\omega_0)v$. Since $Z^{(2)}(i\omega_0)$ is finite, $v \neq 0$ implies that $k \neq 0$. Then, however, v lies in the range of $Z^{(2)}(i\omega_0)$. Combining this fact with the result of 4.46, we see that for $Gv = 0$ it is necessary and sufficient that v lie in the range of $Z^{(2)}(i\omega_0)$: the range of $Z^{(2)}(i\omega_0)$ is exactly the null space of G.

4.48 Now in Halmos[9], par. 37, it is shown that for any dimensionless operator in an n-space the dimensionality of its range space (its *rank*) and the dimensionality of its null space (its *nullity*) add up to n. A similar result and proof hold for operators between **V** and **K**. Let m be the dimensionality of **J**. Then $n - m$ is the rank of $Z^{(2)}(i\omega_0)$, and therefore the dimensionality of the range of $Z^{(2)}(i\omega_0)$, and by 4.47 the dimensionality of the null space of G. Hence, finally,

$$\text{rank }(G) = n - (n - m) = m.$$

By 4.46, therefore, **J** is exactly the range of G.

4.49 Now $\mathbf{N}^{(3)}$, whose admittance matrix is $Y^{(3)}(p)$, might not be ex-

pected to have an impedance matrix. The following reasoning shows that it does have, however:

Consider a $v \epsilon \mathbf{V}$ for which $Y^{(3)}(p)v \equiv 0$. Then from the right side of (9), with (5),

$$v = \frac{2p}{p^2 + \omega_0^2} Z(p)Gv + \frac{2}{p^2 + \omega_0^2} AGv + \frac{2p^2}{p^2 + \omega_0^2} BGv. \tag{10}$$

We have by hypothesis that $Z(p)$ is finite on $p = i\omega$. Therefore we may calculate, by letting $p \to 0$ in (10), that

$$v = \frac{2}{\omega_0^2} AGv,$$

and, by letting $p \to \infty$ in (10), that

$$v = 2BGv.$$

These two equations exhibit v as an element in the range of A and also an element in the range of B. The only possible such v is $v = 0$, by 4.43. Therefore there is no non-zero v such that $Y^{(3)}(p)v \equiv 0$. Then $Z^{(3)}(p) = Y^{(3)}(p)^{-1}$ exists as a PR operator.

4.491 Let

$$L(p) = \frac{1}{p} H + pF \tag{11}$$

be the matrix whose poles at $p = 0$ and $p = \infty$ are those of $Z^{(3)}(p)$. That is, let

$$Z^{(3)}(p) = L(p) + Z^{(4)}(p), \tag{12}$$

where $Z^{(4)}(p)$ is PR and finite at 0 and ∞. Because $Z^{(3)}(p)$ is PR, H and F are both real, symmetric, and semidefinite. Let $\mathbf{N}_L$ be the $2n$-pole whose impedance matrix is $L(p)$, and $\mathbf{N}^{(4)}$ the $2n$-pole with matrix $Z^{(4)}(p)$. In fact, $\mathbf{N}_L$ is realizable. $\mathbf{N}^{(3)}$ is the series combination of $\mathbf{N}_L$ and $\mathbf{N}^{(4)}$, by (12).

4.5 Equations (5), (7), (8), and (12) above are statements about matrices in a particular coordinate frame—that frame appropriate to the given $\mathbf{N}$. We can interpret them as operator relations by simple decree. We wish now to draw a circuit diagram illustrating these relations. To do so, we introduce a suitable new coordinate frame.

Because $G(p)$ is PR and of rank m, we know that a frame can be found in which the matrix for $G(p)$ is an $m \times m$ non-singular matrix bordered by zeros (2.08, or (I, 16.8)). By (7) and the result of 4.48, we

may take the first m current vectors, $k_1, k_2, \cdots, k_m$, specifying this frame, to span **J**. It follows from the matrix form of G then that the corresponding dual vectors $v_1, \cdots, v_m$ span the range of G—i.e., the null space of $Z^{(2)}(i\omega_0)$. We shall adopt such a frame for the further discussion.

Let $\mathbf{K}_1$ be the space spanned by $k_{m+1}, \cdots, k_n$, and $\mathbf{V}_1$ that spanned by $v_{m+1}, \cdots, v_n$, in this frame. Then

$$\begin{aligned} \mathbf{K} &= \mathbf{J} \oplus \mathbf{K}_1 \\ \mathbf{V} &= \mathbf{U} \oplus \mathbf{V}_1, \end{aligned} \tag{1}$$

say, where $\mathbf{U} = \mathbf{J}^*$, $\mathbf{V}_1 = \mathbf{K}_1^*$ [Cf. (I, 10.6)].

If **M** is the name of any given $2n$-pole discussed in the paragraphs 4.4 to date, we let $\overline{\mathbf{M}}$ denote the Cauer equivalent of **M** in this new frame.

4.51 Let $\bar{\mathbf{N}}_G$ be the $2m$-pole whose matrix in the new frame is the $m \times m$ non-singular admittance matrix which, when bordered, gives the matrix of the operator

$$G(p) = \frac{2p}{p^2 + \omega_0^2} G.$$

The $2n$-pole whose matrix is $G(p)$ then obtains by adjoining n-m open circuits to $\bar{\mathbf{N}}_G$. The matrix of $\bar{\mathbf{N}}_G$ operates from **U** to **J** and has an inverse.

4.52 Fig. 5 shows a diagram, which $n = 5$, $m = 3$, of the manner in which we now have **N** represented. The terminals on the extreme left are those of **N**. **N** is obtained from $\bar{\mathbf{N}}$ by a transformer. The horizontal current paths cut the dotted section A-A at points which may be interpreted as the terminals of $\bar{\mathbf{N}}$. Ideal transformers, as in Fig. 1 of I, can be introduced here as needed. Putting them in the diagram merely complicates the picture.

$\bar{\mathbf{N}}$ is the series connection of $\bar{\mathbf{N}}_X$ and $\mathbf{N}^{(2)}$. The terminals of the latter are on B-B. $\mathbf{N}^{(2)}$, again, is the parallel connection of a $2n$-pole obtained from $\bar{\mathbf{N}}_G$ by the adjunction of open circuits, and $\bar{\mathbf{N}}^{(3)}$. The latter has its terminals on C-C. Again, $\bar{\mathbf{N}}^{(3)}$ is the series connection of $\bar{\mathbf{N}}_L$ and $\bar{\mathbf{N}}^{(4)}$.

4.53 Let $\mathbf{M}_{AD}$ be the device between A-A and D-D of Fig. 5. This device has n terminal pairs on A-A and n more on D-D. We may suppose that ideal transformers are attached at each terminal pair as in Fig. 1 of I, since including them in the construction of $\bar{\mathbf{N}}$ would not alter its behavior. Then $\mathbf{M}_{AD}$ is a $2(2n)$-pole.

$\mathbf{M}_{AD}$ is constructed from certain $2r$ poles (with various r) as indicated in the diagram of Fig. 5. The ideal graph* of this diagram (rather, of

* Cf. (I, 4.1).

the relevant part of it between A-A and D-D) obtains from Fig. 5 by inserting ideal branches—two poles—across each terminal pair of each box, and neglecting the outlines of the boxes. The upper m channels of this ideal graph are then T sections, and the lower n-m are degenerate T sections with no shunt arm. This ideal graph is shown in Fig. 6. The ideal branches are shown as small boxes.

The program of the next few paragraphs is to demonstrate that $\mathbf{M}_{AD}$ is a physically realizable $2(2n)$-pole.

4.54 Let us designate the terminal pairs of $\mathbf{M}_{AD}$ on the section A-A by $T_1, T_1', \cdots; T_n, T_n'$, where the r^{th} pair is the intersection with A-A of the leads to the r^{th} terminal pair of $\bar{\mathbf{N}}$. We designate the pairs on

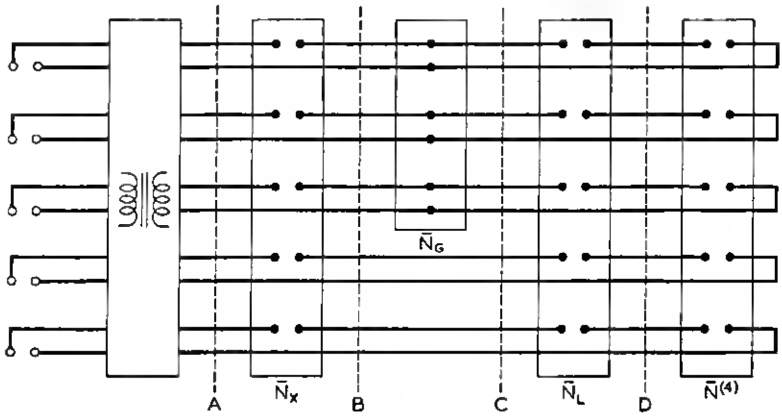


Fig. 5—Original form for $\mathbf{N}$.

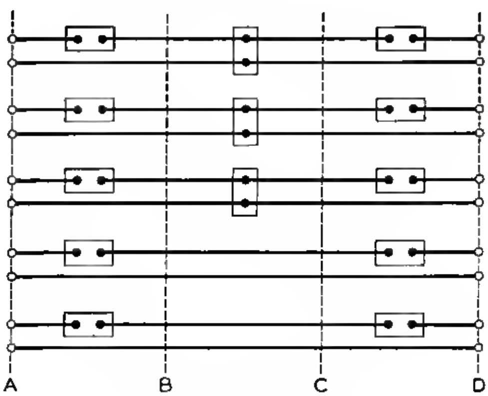


Fig. 6—Ideal graph of $\mathbf{M}_{AD}$.

D-D by $S_1, S_1'; \cdots; S_n, S_n'$, where here the r^{th} pair is the intersection with D-D of the leads to the r^{th} terminal pair of $\bar{\mathbf{N}}^{(4)}$. In each case we orient the pair T, T' or S, S' so that the primed (negative) terminal is on the lead to the primed terminal of $\bar{\mathbf{N}}$ or $\bar{\mathbf{N}}^{(4)}$.

Let the $2n$-tuple

$$[a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n] \tag{2}$$

represent the currents into the terminals of $\mathbf{M}_{\text{AD}}$ in the order

$$T_1, T_2, \cdots, T_n, S_1, \cdots, S_n .$$

Then we may interpret

$$[a_1, \cdots, a_n] \tag{3}$$

as a vector in $\mathbf{K}$ expressed in the coordinate frame introduced for Fig. 5, and also

$$[b_1, \cdots, b_n] \tag{4}$$

as a vector in $\mathbf{K}$ in the same frame. That is, any current vector into $\mathbf{M}_{\text{AD}}$ can be written as an ordered pair

$$k_1, k_2 \tag{5}$$

where each $k_i \in \mathbf{K}$, with the convention that such a pair determines a $2n$-tuple (2) from the n-tuples (3) of k_1 and (4) of k_2.

We shall write the ordered pair (5) in the form

$$k_1 \oplus k_2 . \tag{6}$$

Because we have $\mathbf{K}$ represented in the special way

$$\mathbf{K} = \mathbf{J} \oplus \mathbf{K}_1 ,$$

where $\mathbf{J}$ is the subspace spanned by n-tuples (3) in which the last $n - m$ components vanish (this is (1) of 4.5) we can further split the $2n$-tuple (2) into

$$(j_1 \oplus \ell_1) \oplus (j_2 \oplus \ell_2), \tag{7}$$

where $j_i \in \mathbf{J}$, $\ell_i \in \mathbf{K}_1$, $i = 1, 2$, and in (6)

$$k_i = j_i \oplus \ell_i . \tag{8}$$

Formulas dual to those of (2) through (8) of course hold for voltage $(2n)$-tuples. Let $\mathbf{K}^2$ be the space of current $2n$-tuples (2) (or (7)) and $\mathbf{V}^2$ the space of voltage $(2n)$-tuples

$$[e_1, e_2, \cdots, e_n, f_1, \cdots, f_n] = (u_1 \oplus v_1) \oplus (u_2 \oplus v_2)$$

analogous to (2) and (7), with the scalar product

$$\sum_{r=1}^{n} e_r \bar{a}_r + \sum_{r=1}^{n} f_r \bar{b}_r . \tag{9}$$

It is a common and convenient malpractice in vector algebra to use, for example, the symbol j both for an m-tuple in J and for the n-tuple

$$j \oplus 0 \epsilon \mathbf{J} \oplus \mathbf{K}_1$$

of the form (8). Taking this advantage, we can see that (9) is simply

$$(u_1 + v_1 , j_1 + \ell_1) + (u_2 + v_2 , j_2 + \ell_2) \tag{9'}$$

where here the parentheses denote scalar products between **V** and **K**. The form (9′) can also be derived directly from (1), (7), and (I, 10.6).

4.55 We now wish to compute the voltage-current pairs admitted by $\mathbf{M}_{\text{AD}}$. Referring to Fig. 5, we observe that $\bar{\mathbf{N}}_X$ and $\bar{\mathbf{N}}_L$ both have impedance matrices ($X(p)$ and $L(p)$ respectively, or, rather, the matrix forms of these in the frame of present interest) finite at all p except $p = 0$, $p = \infty$. Each will, therefore, admit any current n-tuple into its terminals, i.e., through its ideal branches, at any but these exceptional frequencies. By construction, $\bar{\mathbf{N}}_G$ has a *non-singular* admittance matrix and therefore also will admit any current m-tuple into its terminals (2.07), except at most at certain isolated frequencies. It is evident by Kirchoff's laws applied to Fig. 6 then that $\mathbf{M}_{\text{AD}}$ will admit any current $2n$-tuple of the form

$$(j_1 \oplus k) \oplus (j_2 \oplus (-k)) \tag{10}$$

where $j_i \epsilon \mathbf{J}$, $i = 1, 2$, and $k \epsilon \mathbf{K}_1$, except at most at finitely many exceptional frequencies. Conversely, if the current $2n$-tuple specified by (7) is that in $\mathbf{M}_{\text{AD}}$, conservation at the absent shunt arms of the lower degenerate T-sections implies that, as elements of **K**,

$$k_1 + k_2 = 0,$$

that is, the current is of the form (10). Hence $2n$-tuples of the form (10) span the space of currents admitted by $\mathbf{M}_{\text{AD}}$. Let us call this space $\mathbf{K}_{\mathbf{M}}^2$. It is a proper subspace of $\mathbf{K}^2$ unless $m = n$.

4.56 Let $G^{-1}(p)$ denote the $m \times m$ impedance matrix of $\bar{\mathbf{N}}_G$. Then by (7) of 4.4, interpreted as an operator equation,

$$G^{-1}(p) = \left(\frac{1}{2}\, p + \frac{\omega_0^2}{2p}\right) G^{-1} \tag{11}$$

where G^{-1} is a real, constant, symmetric, non-singular $m \times m$ matrix.

We can now compute the voltage across $\mathbf{M}_{AD}$ corresponding to the current (10). Let w be the n-tuple of voltages appearing at the section B-B or C-C of Fig. 6, with its components listed in the appropriate order. Then we may interpret w as a vector in $\mathbf{V}$, and write it

$$w = u_0 \oplus v_0 \tag{12}$$

where $u_0 \in \mathbf{U}$, $v_0 \in V_1$. Now by Kirchoff's current law applied to the shunt arms in the upper channels of Fig. 6, the current into $\bar{\mathbf{N}}_G$ is

$$j_1 + j_2,$$

and therefore

$$u_0 = G^{-1}(p)(j_1 + j_2). \tag{13}$$

By Kirchoff's voltage law applied to a typical mesh which begins on A-A, goes through $\mathbf{N}_X$ to B-B, and then through a shunt arm and returns to A-A, the voltage n-tuple appearing at A-A is

$$X(p)(j_1 + k) + w.$$

Referring to (12), let us use u_0 also to denote the vector

$$u_0 \oplus 0 \in \mathbf{V},$$

and v_0 to denote

$$0 \oplus v_0 \in \mathbf{V}.$$

Interpreting (13) in this way we get

$$X(p)(j_1 + k) + G^{-1}(p)(j_1 + j_2) + v_0 \tag{14}$$

as the voltage n-tuple on A-A.

A similar calculation gives

$$L(p)(j_2 - k) + G^{-1}(p)(j_1 + j_2) + v_0 \tag{15}$$

as the voltage n-tuple on D-D. The ordered pair (14), (15) then gives the voltage $2n$-tuple corresponding to (10), in the notation analogous to (5).

4.57 $X(p)$, $L(p)$, and $G^{-1}(p)$, respectively, are defined in (6) of 4.44, (11) of 4.491, and (11) of 4.56. Each one is finite except at $p = 0$ and $p = \infty$. Let $\Gamma_{\mathbf{M}}$ be the complex plane from which these two points are deleted. It is now possible to show that the linear correspondence whose pairs, for each $p \in \Gamma_{\mathbf{M}}$, are the voltages (14), (15) $\in \mathbf{V}^2$ and the currents (10) $\in \mathbf{K}^2$, satisfies P1 through P7 of (I, 6, 7)—that is, is PR (I, 16.71).

In the present special circumstances it is almost as easy to study $\mathbf{M}_{AD}$ in a slightly different way than this. Since fewer direct references to I are involved, we shall take the alternative path.

We first calculate the scalar product between the voltage (14), (15) and an arbitrary current of the form (10), say the current

$$(h_1 \oplus \ell) \oplus (h_2 \oplus (-\ell)) \in \mathbf{K}_\mathbf{M}^2 .$$

To do so, we consider the form (9′) for such a product. In the first writing, then, this scalar product is

$$(X(p)(j_1 + k) + G^{-1}(p)(j_1 + j_2) + v_0 , h_1 + \ell) \\ + (L(p)(j_2 - k) + G^{-1}(p)(j_1 + j_2) + v_0 , h_2 - \ell).$$

Each of these scalar products has three voltages appearing in it. Distributing the products over these voltages, and using the facts that the range of $G^{-1}(p)$ is $\mathbf{J}$ and that $v_0 \in \mathbf{V}_1 = (\mathbf{J})^0$ we get a second form:

$$(X(p)(j_1 + k), h_1 + \ell) + (G^{-1}(p)(j_1 + j_2), h_1) + (v_0 , \ell) \\ + (L(p)(j_2 - k), h_2 - \ell) + (G^{-1}(p)(j_1 + j_2), h_2) + (v_0 , - \ell).$$

The terms involving v_0 go out and we can collect to

$$(X(p)(j_1 + k), h_1 + \ell) + (G^{-1}(p)(j_1 + j_2), h_1 + h_2) \\ + (L(p)(j_2 - k), h_2 - \ell). \tag{16}$$

This is the desired scalar product.

4.58 Let us now consider the $(n + m)$-tuples

$$[a_1 , a_2 , \cdots , a_n , b_1 , \cdots , b_m] = j_1 \oplus k \oplus j_2 \tag{17}$$

obtained from (2) by deleting the $b_{m+1} , \cdots , b_n$. We still interpret these as currents into the relevant terminals of $\mathbf{M}_{AD}$. We also observe that when the current (17) is given, (2) can be determined, because by (10)

$$a_{m+s} + b_{m+s} = 0, \qquad s = 1, 2, \cdots , n - m.$$

Given (17), and therefore (2) or (10), we can determine the voltages (14) and (15), where v_0 is an arbitrary element of $\mathbf{V}_1$. Let us agree now always so to choose v_0 that the component of (15) in the subspace $\mathbf{V}_1$ vanishes. This means that, in (17), we have specified arbitrarily the currents into the left-hand terminals of $\mathbf{M}_{AD}$ (on A-A) and into the upper m of the right-hand terminals. We have also agreed that the voltages across the lower n-m terminals on D-D shall be zero, so that (15) is an

n-tuple of the form

$$u \oplus 0 \tag{18}$$

where $u \in \mathbf{U}$. Regarding (15), with this determination of v_0, as simply an m-tuple u (ignoring its last $n - m$ zero components), we see that (17) and the ordered pair (14), (15) are now currents and voltages in a $2(n + m)$-pole $\mathbf{M}_{\mathrm{AD}}^*$ obtained from $\mathbf{M}_{\mathrm{AD}}$ by shorting and thereafter ignoring the lower $n - m$ terminals on D-D.

4.59 Now (17) is unrestricted. Given it, the corresponding voltages can be computed from (14) and (15) by determining v_0 so that (15) lies in $\mathbf{U}$. Hence $\mathbf{M}_{\mathrm{AD}}^*$ has an impedance matrix, since any single valued linear mapping from (17) to voltages can be described by a matrix. Our job is now to show that this matrix comes under 3.1. Before doing this, however, we shall point out that a realization of $\mathbf{M}_{\mathrm{AD}}^*$ provides one for $\mathbf{M}_{\mathrm{AD}}$.

Fig. 7 shows how a $2(2n)$-pole equivalent to $\mathbf{M}_{\mathrm{AD}}$ would be constructed from $\mathbf{M}_{\mathrm{AD}}^*$. The equivalence is evident almost at once: The pairs of $\mathbf{M}_{\mathrm{AD}}^*$ are the currents (17) and the voltages (14) and (15) with a special determination of v_0, where (15) is regarded as an m-tuple. The current (10) is clearly that which flows in the $2(2n)$-pole of Fig. 7 when (17) flows in $\mathbf{M}_{\mathrm{AD}}^*$. Furthermore, regarding (15) as an n-tuple of the form (18), we see that the voltages in Fig. 7 can be obtained from (14), (15) by adding an arbitrary voltage of the form

$$(0 \oplus v) \oplus (0 \oplus v),$$

where $v \in \mathbf{V}_1$ of course. This arbitrary added voltage eliminates the special role played by v_0 in (14) and (15). Hence therein v_0 itself may be considered to be an arbitrary element of $\mathbf{V}_1$, and (14), (15) represent the voltages in Fig. 7. The pairs admitted by the $2(2n)$-pole of Fig. 7 are therefore exactly those admitted by $\mathbf{M}_{\mathrm{AD}}$, Q.E.D.

4.60 We have now established that $\mathbf{M}_{\mathrm{AD}}^*$ has an impedance matrix, say $M(p)$. $M(p)$ operates from an $(n + m)$ space of currents (17) of 4.58 to an $(n + m)$ space of voltages (14), (15) of 4.56, where in (15) we properly choose v_0 so that the last $(n - m)$ components are zero and can be ignored.

Now any impedance matrix $\hat{Z}(p)$ is completely determined when we know for each two currents m_1 and m_2 the scalar product

$$(\hat{Z}(p)m_1, m_2) \tag{1}$$

(Cf. Halmos[9], par. 53). We shall make this computation for $M(p)$. The

currents (17) of 4.58 may be regarded as elements of the subspace (10) of 4.55. We have called this subspace $\mathbf{K}_\mathbf{M}^2$. The voltages (14), (15), with v_0 chosen to make (15) an n-tuple of the form (18) (4.58), are elements of a subspace $\mathbf{V}_\mathbf{M}^2$ of $\mathbf{V}^2$.

It is evident at once that the scalar product between a current $(n + m)$-tuple (17) and the $(n + m)$-tuple (14), (15) (v_0 properly chosen!) is exactly the same as the scalar product between the current $(2n)$-tuple (10) and the $(2n)$-tuple formed from the $(n + m)$-tuple (14), (15) by adjoining $(n - m)$ zeros to expand (15) to an n-tuple of the form (18).

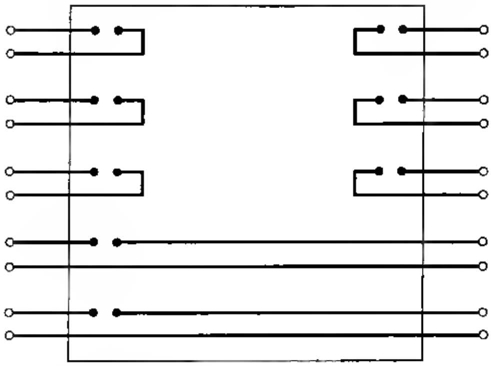


Fig. 7 = Construction of $\mathbf{M}_{\mathbf{AD}}$ from $\mathbf{M}_{\mathbf{AD}}$*. The solid terminals are those of $\mathbf{M}_{\mathbf{AD}}$*, the open circles those of $\mathbf{M}_{\mathbf{AD}}$.

Now we know that we may regard (15) as an n-tuple of the form (18) by a suitable choice of v_0 . But we calculated in 4.57 the scalar product between an arbitrary $(2n)$-tuple and (14), (15) with an arbitrary v_0 . The answer was (16) of 4.57. By proper choice of v_0 , then, (16) represents the bilinear form (1) above for $M(p)$. Since (16) is independent* of v_0 , it contains in itself the whole of the properties of $M(p)$.

4.61 To show that $M(p)$ is PR, we need show only that $M(p)$ is symmetric and that is *quadratic* form ($j_i = h_i$ and $k = \ell$ in (16)) is a PR function of p (2.09).

By their definitions, $X(p)$, $L(p)$, and $G^{-1}(p)$ are all symmetric. Hence if all currents are real, the value of (16) is unchanged by interchanging j_i with h_i , $i = 1, 2$, and k with ℓ. Therefore $M(p)$ is symmetric.

4.62 Henceforth we consider the quadratic from

* This is the gist of P3 of (I, 7.4). Use of the results of I here would have given a more direct but much less constructive representation of $\mathbf{M}_{\mathbf{AD}}$.

$$(X(p)(j_1 + k), j_1 + k) + (G^{-1}(p)(j_1 + j_2), j_1 + j_2) \\ + (L(p)(j_2 - k), j_2 - k) \tag{2}$$

obtained from (16). By the definitions of $X(p)$, $L(p)$, and $G^{-1}(p)$ this is a rational function taking real values for real p. Hence we need only show of (2) that its real part is non-negative when $Re(p) > 0$ to show that it and $M(p)$ are PR.

Referring to (6) and (11) of paragraph 4.4 and (11) of 4.56 for the definitions, we see that (2) can be written

$$\begin{aligned}\frac{1}{p}\Big[&-(A(j_1 + k), j_1 + k) + \frac{\omega_0^2}{2}(G^{-1}(j_1 + j_2), j_1 + j_2) \\ &+ (H(j_2 - k), j_2 - k)\Big] \\ + p\Big[&-(B(j_1 + k), j_1 + k) + \frac{1}{2}(G^{-1}(j_1 + j_2), j_1 + j_2) \\ &+ (F(j_2 - k), j_2 - k)\Big].\end{aligned} \tag{3}$$

That is, the quadratic form in question has poles, simple ones, only at 0 and ∞, and has no constant term. If we can show that the residues at these poles are non-negative, then it will follow not only that $M(p)$ is PR but that $M(p)$ is of the form

$$\frac{1}{p} M_0 + pM_\infty$$

where each of these summands is realizable by 3.1.

Unfortunately, there still remains some computation to verify that the residues of (3) are non-negative.

4.62 We first recapitulate some relations obtained earlier:

$$Z^{(2)}(p) = Z(p) + \frac{1}{p} A + pB; \tag{4}$$

this is (5) of 4.42.

$$Y^{(2)}(p) = \frac{2p}{p^2 + \omega_0^2} G + Y^{(3)}(p); \tag{5}$$

this is (7) and (8) of 4.45.

$$Z^{(3)}(p) = \frac{1}{p} H + pF + Z^{(4)}(p); \tag{6}$$

this is (11) and (12) of 4.491.

By their definitions,

$$Z^{(i)}(p) = [Y^{(i)}(p)]^{-1}$$

for $i = 2, 3$. By hypothesis, $Z(p)$ and

$$Z(p)^{-1} = Y(p)$$

are both finite everywhere on $p = i\omega$. By its construction, $Z^{(4)}(p)$ is finite at $p = 0$ and $p = \infty$.

4.63 We claim now that each $Y^{(i)}(p)$ is finite at $p = 0$ and ∞, $i = 2, 3$.

Proof: We need consider only $Y^{(2)}(p)$ since $Y^{(3)}(p)$ differs from it by something which vanishes at $p = 0$ and $p = \infty$ ((5) above). Let

$$Y^{(2)}(p) = \tilde{Y}(p) + \frac{1}{p} E + pQ$$

where $\tilde{Y}(p)$ is finite at $p = 0$ and $p = \infty$. Since $Y^{(2)}(p)$ is PR (4.43), E and Q are real and symmetric.

Using the form (4) above for $Z^{(2)}(p)$,

$$\begin{aligned} 1 = Z^{(2)}(p)Y^{(2)}(p) = {} & Z(p)\tilde{Y}(p) + BE + AQ \\ & + p(Z(p)Q + B\tilde{Y}(p)) + p^2BQ \\ & + \frac{1}{p}(Z(p)E + A\tilde{Y}(p)) + \frac{1}{p^2} AE. \end{aligned} \tag{7}$$

Multiplying through by $p^2, p, \frac{1}{p^2}, \frac{1}{p}$ and taking limits as $p \to 0, 0, \infty, \infty$, respectively, we obtain

$$\begin{aligned} AE &= 0 \\ Z(0)E + A\tilde{Y}(0) &= 0, \\ BQ &= 0, \\ Z(\infty)Q + B\tilde{Y}(\infty) &= 0. \end{aligned} \tag{8}$$

We can also write a formula like (7) with the factors in reverse order, and obtain the analogous forms to (8) in which the factors are commuted. Let us call these commuted relations (8′). Multiply the second relation (8) on the left by E and use the first relation of (8′). We obtain

$$EZ(0)E = 0. \tag{9}$$

Working similarly with the last two relations in (8) and (8′), we get

$$QZ(\infty)Q = 0. \tag{10}$$

Now let v be an arbitrary voltage in $\mathbf{V}$ and let

$$w = Z(0)Ev.$$

Then $w \in \mathbf{V}$, and by (9) the current

$$Ew = 0$$

for any v. Hence

$$\begin{aligned} 0 &= (v, Ew) = \overline{(w, E^*v)} = (\bar{w}, \bar{E}^*\bar{v}) \\ &= (\bar{Z}(0)\bar{E}\bar{v}, E'\bar{v}) \end{aligned} \tag{11}$$

by (I, 7.2, 14.0). Now E is real and symmetric, as noted above. Hence $E = \bar{E} = E'$. Furthermore, $Z(0)$ is real, so (11) becomes

$$(Z(0)Eu, Eu) = 0 \tag{12}$$

where $u = \bar{v}$ is any element of $\mathbf{V}$. Now $Z(p)$ is non-singular on $p = i\omega$, and its real part is semidefinite there. At $p = 0$, $Z(0)$ is its own real part, hence semidefinite and non-singular, hence definite. Then (12) implies that $Eu = 0$. This being true for all $u \in \mathbf{V}$, $E = 0$.

The proof that $Q = 0$ follows similarly from (10).

4.64 With $Y^{(2)}(p)$ and $Y^{(3)}(p)$ simplified at $p = 0$ and ∞, we can go back and compute

$$\begin{aligned} 1 &= Z^{(2)}(p)Y^{(2)}(p) \\ &= \left(Z(p) + \frac{1}{p}A + pB\right)\left(\frac{2p}{p^2 + \omega_0^2}G + Y^{(3)}(p)\right). \end{aligned} \tag{13}$$

Of the six terms obtained on expanding this exactly one, namely

$$\frac{1}{p}AY^{(3)}(p)$$

is not obviously finite at $p = 0$, and another,

$$pBY^{(3)}(p)$$

is not *a priori* finite at $p = \infty$. We conclude by multiplying through by p and letting $p \to 0$, and dually at $p = \infty$, that

$$\begin{aligned} AY^{(3)}(0) &= 0 = Y^{(3)}(0)A \\ BY^{(3)}(\infty) &= 0 = Y^{(3)}(\infty)B, \end{aligned} \tag{14}$$

where the commuted form can be established by a new calculation from $1 = Y^2(p)Z^2(p)$, or by taking transposes.

In a similar way, we compute from

$$1 = Z^{(3)}(p)Y^{(3)}(p) = Z^{(4)}(p)Y^{(3)}(p) + \frac{1}{p} HY^{(3)}(p) + pFY^{(3)}(p) \quad (15)$$

that

$$\begin{aligned} HY^{(3)}(0) &= 0 = Y^{(3)}(0)H, \\ FY^{(3)}(\infty) &= 0 = Y^{(3)}(\infty)F. \end{aligned} \quad (16)$$

Now $Y^{(3)}(p)$ is finite at 0 and ∞, so we may expand it in a power series about either point. Let these be

$$\begin{aligned} Y^{(3)}(p) &= Y^{(3)}(0) + pY_1^{(3)}(0) + O(p^2), \\ Y^{(3)}(p) &= Y^{(3)}(\infty) + \frac{1}{p} Y_1^{(3)}(\infty) + O\left(\frac{1}{p^2}\right). \end{aligned} \quad (17)$$

Putting the appropriate one of these into (13) and taking a limit at 0 or ∞ we get, by using (14), that

$$\begin{aligned} 1 &= \frac{1}{\omega_0^2} AG + AY_1^{(3)}(0) + Z(0)Y^{(3)}(0), \\ 1 &= 2BG + BY_1^{(3)}(\infty) + Z(\infty)Y^{(3)}(\infty). \end{aligned} \quad (18)$$

A relation (18′) with factors commuted is also true.

We may also put (17) into (15) and get

$$\begin{aligned} 1 &= Z^{(4)}(0)Y^{(3)}(0) + HY_1^{(3)}(0), \\ 1 &= Z^{(4)}(\infty)Y^{(3)}(\infty) + FY_1^{(3)}(\infty), \end{aligned} \quad (19)$$

and also a commuted form (19′).

Right multiply the first line of (19) by A and the second by B, and use (14). This gives

$$\begin{aligned} A &= HY_1^{(3)}(0)A, \\ B &= FY_1^{(3)}(\infty)B. \end{aligned} \quad (20)$$

Left multiply the first line of (18′) by H and the second by F. This gives, by (16),

$$\begin{aligned} H &= \frac{2}{\omega_0^2} HGA + HY_1^{(3)}(0)A, \\ F &= 2FGB + FY_1^{(3)}(\infty)B. \end{aligned} \quad (21)$$

Using (20) in (21), we have the relations

$$\frac{2}{\omega_0^2} HGA = H - A,$$
$$2FGB = F - B. \tag{22}$$

These are fundamental to the evaluation of the residues of (3). Before calculating these residues, we draw a further important conclusion from the formulas just developed.

Relation (20) exhibits A as a product of H and a possibly singular matrix (viz., $Y_1^{(3)}(0)A$). Hence

$$\text{rank } (A) \leq \text{rank } (H).$$

But relation (21) shows H as a product of A by

$$\frac{2}{\omega_0^2} HG + HY_1^{(3)}(0).$$

Hence

$$\text{rank } (H) \leq \text{rank } (A).$$

That is,

$$\text{rank } (A) = \text{rank } (H),$$
$$\text{rank } (B) = \text{rank } (F), \tag{23}$$

the latter being established in the same way.

4.65 The formulas developed in 4.64 are all quite symmetric as between relations obtained at $p = \infty$ and those at $p = 0$. We shall now continue to the evaluation of the residue of (3) at $p = \infty$. The evaluation at $p = 0$ proceeds in an exactly similar manner.

The residue in question is, from (3),

$$-(B(j_1 + k), j_1 + k) + \tfrac{1}{2}(G^{-1}(j_1 + j_2), j_1 + j_2) + (F(j_2 - k), j_2 - k). \tag{24}$$

Here j_1 and j_2 are any elements of $\mathbf{J}$ and k any element of $\mathbf{K}_1$. The range of G is $\mathbf{J}$ and the operator G^{-1} operates from $\mathbf{J}$ to $\mathbf{U} = \mathbf{J}^*$, representing the inverse to the operation G from $\mathbf{U}$ to $\mathbf{J}$.

Let us define h and eliminate j_2 by the relation

$$j_2 = 2h + 2GB(j_1 + k) - j_1 . \tag{25}$$

Since the range of G is $\mathbf{J}$, $h \in \mathbf{J}$.

The definition analogous to (25) for the other pole of (3) is

$$j_2 = \frac{2}{\omega_0^2} h + \frac{2}{\omega_0^2} GA(j_1 + k) - j_1 .$$

We shall now say no more about this pole.

Putting (25) into (24) we get at once the form

$$-(B(j_1 + k), j_1 + k) + (G^{-1}h + G^{-1}GB(j_1 + k), 2h + 2GB(j_1 + k))$$
$$+ (2Fh + 2FGB(j_1 + k) - Fj_1 - Fk, 2h + 2GB(j_1 + k) - j_1 - k).$$

Here we cannot at once put $G^{-1}G = 1$, because this is only true in $\mathbf{U}$. We expand in the following way: The first product is left intact, the second is expanded by distributivity into four terms, and in the third we use (22) and expand into five terms by distributivity. The ten resulting terms are:

$$\begin{aligned}
&- (B(j_1 + k), j_1 + k) + 2(G^{-1}h, h) \\
&+ 2(G^{-1}GB(j_1 + k), h) + 2(G^{-1}h, GB(j_1 + k)) \\
&+ 2(G^{-1}GB(j_1 + k), GB(j_1 + k)) \\
&+ 4(Fh, h) - 2(B(j_1 + k), h) \\
&+ 2(Fh, 2GB(j_1 + k) - j_1 - k) \\
&- 2(B(j_1 + k), GB(j_1 + k)) + (B(j_1 + k), j_1 + k).
\end{aligned}$$

Enumerate these terms 1, 2, $\cdots$, 10 in the order written. We shall show by combining that only 2 and 6 remain.

Clearly 1 and 10 cancel.

Consider the operator $G^{-1}G$ as we have defined it. If $v \in \mathbf{V}$, we can put

$$v = u + v_1$$

where $u \in \mathbf{U}$, $v_1 \in \mathbf{V}_1$. Then

$$Gv = Gu + Gv_1 = Gu,$$

because of the matrix form for G in the coordinate system chosen in 4.5. By definition of G^{-1} (in 4.56), since $u \in \mathbf{U}$,

$$G^{-1}Gu = u.$$

Hence, combining the last three relations,

$$G^{-1}Gv = v - v_1 \tag{26}$$

for any $v \in \mathbf{V}$, where v_1 is a suitable element of $\mathbf{V}_1$ (depending on v of course).

Using (26) in term 3, we get for this term

$$2(B(j_1 + k), h) - 2(v_1, h)$$

for some $v_1 \in \mathbf{V}_1$. But $h \in \mathbf{J} = (\mathbf{V}_1)^0$ ((1) of 4.5). Hence the second term here vanishes and term 3 cancels term 7. By an exactly similar argument, since $GB(j_1 + k) \in \mathbf{J}$, we find that term 5 cancels term 9.

Consider term 4, and write it in the form

$$2(G^{-1}h, k_1) = \overline{2((G^{-1})^* k_1, h)}$$
$$= 2((\bar{G}^{-1})^* \bar{k}_1, \bar{h}) = 2(G^{-1}\bar{k}_1, \bar{h}).$$

This follows by (I, 7.2, 14.0) and the fact that G^{-1} is symmetric. Putting in the definition of k_1, and using the fact that G and B are real, we get

$$2(G^{-1}\bar{k}_1, \bar{h}) = 2(G^{-1}\overline{GB}(\bar{j}_1 + \bar{k}), \bar{h})$$
$$= 2(G^{-1}GB(\bar{j}_1 + \bar{k}), \bar{h}).$$

Now $\mathbf{J}$ is real (4.42) so $\bar{h} \in \mathbf{J}$. Therefore the reasoning used on term 3 yields finally

$$2(B(\bar{j}_1 + \bar{k}), \bar{h})$$

as the value of term 4.

We now write term 8 as

$$2(Fh, k_2)$$

and transform it to

$$2(F\bar{k}_2, \bar{h}),$$

by the reasoning just used on 4. Putting in what k_2 is, this is

$$2(2F\overline{GB}(\bar{j}_1 + \bar{k}) - F\bar{j}_1 - F\bar{k}, \bar{h}).$$

Using the reality of G and B, and (22), this is

$$-2(B(\bar{j}_1 + \bar{k}), \bar{h}).$$

This cancels term 4 and all terms save 2 and 6 are accounted for. Finally, then, the residue of (3) at $p = \infty$ is

$$2(G^{-1}h, h) + 4(Fh, h). \tag{27}$$

Since G^{-1} is definite in $\mathbf{J}$ and F is semidefinite, this residue is non-negative, and indeed not zero if $h \neq 0$ and $h \in \mathbf{J}$.

4.7 We have established the non-negativity of the residue of (3) at $p = \infty$. A similar argument (exactly parallel, in fact) will establish the same for the residue at $p = 0$. Each term in the representation

$$M(p) = \frac{1}{p} M_0 + pM_\infty$$

of 4.61 is then realizable by 3.1. Hence $\mathbf{M}_{\mathrm{AD}}^*$ is a realizable reactance $2(n + m)$-pole, and so therefore is $\mathbf{M}_{\mathrm{AD}}$, as we noted in discussing Figure 7 (4.59). Therefore, if $\bar{\mathbf{N}}^{(4)}$ of Figure 5 is physically realizable, so also is $\bar{\mathbf{N}}$ and therefore $\mathbf{N}$. We denote by $\mathbf{N}_1$ the $\bar{\mathbf{N}}^{(4)}$ obtained in this way from $\mathbf{N}$, and define IB as the operation which constructs $\mathbf{N}_1$ from $\mathbf{N}$.

We must still establish the claims made in 4.04 for IB. No properties of $\bar{\mathbf{N}}^{(4)} = \mathbf{N}_1$ have been proved beyond the existence of its impedance matrix, $Z^{(4)}(p)$, but this is all that is claimed in the third column of 4.04. The fifth column is also established. We must now however compare the degree of $\mathbf{N}_1$, i.e., of $Z^{(4)}(p)$, with that of $Z(p)$.

By 2.13, 2.14 and 2.15 applied to (4), (5), and (6) of 4.62,

$$\delta(Z^{(2)}) = \delta(Z) + \text{rank }(A) + \text{rank }(B),$$

$$\delta(Z^{(2)}) = \delta(Y^{(2)}) = \delta(Y^{(3)}) + 2 \text{ rank }(G),$$

$$\delta(Y^{(3)}) = \delta(Z^{(3)}) = \delta(Z^{(4)}) + \text{rank }(H) + \text{rank }(F).$$

We know $m = \text{rank }(G) \geq 1$. Let

$$r = \text{rank }(A) + \text{rank }(B).$$

Then from (23), and the relations above in order,

$$\begin{aligned} \delta(Z) = \delta(Z^{(2)}) - r &= (\delta(Z^{(3)}) + 2m) - r \\ &= (\delta(Z^{(4)}) + r) + 2m - r \\ &= \delta(Z^{(4)}) + 2m. \end{aligned}$$

Hence $\delta(Z) - \delta(Z^{(4)}) = \delta(\mathbf{N}) - \delta(\mathbf{N}_1) = 2m > 0$. The claims of 4.04 are then established.

4.71 We must yet verify 4.07 for IB. Let $\delta(M)$ be the degree of

$$M(p) = \frac{1}{p} M_0 + pM_\infty.$$

Then by 3.21, $\mathbf{M}_{\mathrm{AD}}^*$, whose matrix is $M(p)$, can be realized with $\delta(M)$ reactive elements. By Figure 7, then $\mathbf{M}_{\mathrm{AD}}$ can be so realized, and it follows that exactly $\delta(M)$ reactive elements are comprised between $\mathbf{N}$ and $\mathbf{N}_1$ under IB.

Now by 2.14 and 2.15,

$$\delta(M) = \text{rank }(M_0) + \text{rank }(M_\infty).$$

We shall compute the second term. The first is obtained by an exactly parallel calculation.

Using the fact that $M(p)$ is determined by its quadratic form, we see that M_∞ is the matrix whose form is the residue of that of $M(p)$ at $p = \infty$. This residue was computed in (27) of 4.65 to be

$$2(G^{-1}h, h) + 4(Fh, h) \tag{1}$$

when the current vector, (17) of 4.58, is

$$j_1 \oplus k \oplus j_2, \tag{2}$$

and, (25) of 4.65,

$$2h = j_1 + j_2 - 2GB(j_1 + k). \tag{3}$$

Here $j_1, j_2 \in \mathbf{J}$ and $k \in \mathbf{K}_1$.

Now M_∞ is an $(n + m) \times (n + m)$ matrix by construction. Then

$$\nu = n + m - \text{rank}\,(M_\infty) \tag{4}$$

is its nullity, the dimension of its null space. This is proved in Halmos[9], par. 37, for dimensionless operators, and a similar proof applies to impedance operators.

Now for any symmetric and semidefinite impedance operator $\hat{Z}$, the null space of $\hat{Z}$ is exactly the aggregate of currents k such that the quadratic form

$$(\hat{Z}k, k) = 0.$$

This may be seen at once by choosing a coordinate frame in which the matrix of $\hat{Z}$ is diagonal. Since we know from 4.65 that M_∞ is symmetric and semidefinite, we can compute ν as the dimensionality of the space of vectors (2) above for which (1) vanishes.

As noted in 4.65, $h \in \mathbf{J}$, and (1) vanishes if and only if $h = 0$, because G^{-1}, as an operator from $\mathbf{J}$ to $\mathbf{U}$, is definite (semidefinite and non-singular). Hence ν is the maximum number of linearly independent vectors (2) for which, from (3),

$$(1 - 2GB)j_1 + j_2 - 2GBk = 0. \tag{5}$$

The left member of (5) is a vector in $\mathbf{J}$ depending linearly and homogeneously on the vector (2). Hence, regarding $\mathbf{J}$ as a subspace of the space $\mathbf{J} \oplus \mathbf{K}_1 \oplus \mathbf{J}$ in which (2) lies, the left member of (5) is the value in $\mathbf{J} \oplus \mathbf{K}_1 \oplus \mathbf{J}$ of a certain linear operation applied to the vector (2). Let us call this operator P. The number ν, by definition the number of linearly independent vectors (2) for which (5) holds, is the nullity of P. The dimension of P is $n + m$, and its rank is clearly m because the left member of (5)—a typical element in the range of P—lies in $\mathbf{J}$ and by

suitable choice of j_2 can be made to be any element of **J**. Hence the nullity of P is $(n + m) - m = n$ (Halmos[9], par. 37). That is

$$\nu = n,$$

and, by (4)

$$\text{rank } (M_\infty) = m.$$

A parallel argument will establish the same result for M_0. Hence

$$\delta(M) = 2m = \delta(\mathbf{N}) - \delta(\mathbf{N}_1)$$

by a result of 4.7. Therefore $\mathbf{M}_{\text{AD}}^*$ and $\mathbf{M}_{\text{AD}}$ can be realized with

$$\delta(\mathbf{N}) - \delta(\mathbf{N}_1)$$

reactive elements and 4.07 holds for IB.

V. THE DEGREE OF A RATIONAL MATRIX

5.0 In this section we consider arbitrary $n \times n$ matrices $Z(p)$ whose elements are rational functions of the complex variable p. They are treated, generally, as arrays of functions with certain rules for addition, multiplication, and reciprocation, without geometric interpretation. A geometric development is possible, but would be cumbrous. Related ideas may be found, geometrically developed, in Appendix I of Halmos[9].

This section deals wholly with concepts well known in the algebraic theory of matrices over an arbitrary field—in this case the field of rational functions. I have not found, however, any place where the particular developments which seem to be needed here are made sufficiently explicitly for reference. Accordingly, the presentation here is somewhat detailed. The particular path of argument followed is only one of many possible; it was chosen to lead easily to results needed in Section 6, and to parallel generally the rest of the paper.

This section could be abbreviated somewhat if one restricted himself to PR matrices $Z(p)$. We prefer not to limit the applicability of these results, however, since they may well be useful in non-passive realizability theory.

5.01 *Definition:* If $R(p)$ is a rational function of the form

$$R(p) = (p - p_0)^m R_1(p),$$

where $R_1(p)$ is finite and not zero at p_0, and m may be of any sign, we call m the exponent of $(p - p_0)$ in $R(p)$. The number

$$r = \sup(-m, 0)$$

is called the order of the pole of $R(p)$ at p_0, even if $r = 0$.

5.1 Let $Z(p)$ be an $n \times n$ matrix whose elements $Z_{rs}(p)$ are rational functions of the complex variable p. We write

$$Z_{rs}(p) = \frac{N_{rs}(p)}{D_{rs}(p)},$$

where N_{rs} and D_{rs} are relatively prime polynomials. Let $\Psi_Z(p)$ be the least common multiple of all $D_{rs}(p)$, $(1 \leq r, s \leq n)$, so normalized that the coefficient of the highest power of p appearing in $\Psi_Z(p)$, (the *leading coefficient*) is unity. Then $\Psi_Z(p)$ is uniquely determined by $Z(p)$.

The matrix $\Psi_Z(p)Z(p)$ has polynomial elements. Its *Smith normal form* is a diagonal matrix $E(p)$,

$$E(p) = \begin{bmatrix} E_1(p) & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & E_2(p) & & & & \\ \vdots & & \ddots & & & \\ \vdots & & & E_R(p) & & \\ 0 & & & & 0 \cdot & \\ & & & & & \cdot 0 \end{bmatrix} = A(p)\Psi_Z(p)Z(p)B(p), \quad (1)$$

with the following properties:

(a) R is the rank of $\Psi_Z(p)Z(p)$.

(b) Each $E_i(p)$, $1 \leq i \leq R$, is a polynomial with unit leading coefficient.

(c) Each $E_i(p)$ is a factor of $E_{i+1}(p)$, $1 \leq i \leq R - 1$.

(d) $A(p)$ and $B(p)$ are polynomial matrices, each with a constant non-vanishing determinant.

(e) $E_1(p)E_2(p) \cdots E_k(p)$ is the normalized (and therefore unique) highest common factor of all k-rowed minor determinants of $\Psi_Z(p)Z(p)$.

These properties of $E(p)$ are developed for example, in Bocher[15], Theorems 2 and 3 of paragraph 91 and Theorem 1 of paragraph 94. A simple variation of this last cited theorem will also prove the following uniqueness lemma.

5.11 *Lemma:* If some $E^0(p)$ satisfies (1) and (a), (b), (c) and (d) above, all written with superscripts on each E, and on A and B, then $E^0(p) = E(p)$.

Proof: $E^0(p)$ is equivalent to $E(p)$ in the sense of paragraph 94 of

Bocher[15], for

$$E^0(p) = A^0(p)A^{-1}(p)E(p)B^{-1}(p)B^0(p).$$

Therefore it is also equivalent in the sense of par. 91 of Bocher[15], (for this is Theorem 1 of paragraph 94). Hence the normalized greatest common factor of all k-rowed minors of $E^0(p)$ is the same as that of $E(p)$, that is, $E_1(p) \cdots E_k(p)$. But the greatest common factor of all k rowed minors of $E^0(p)$ is $E_1^0(p) \cdots E_k^0(p)$, because of property (c). In particular then $E_1(p) = E_1^0(p)$, and consequently $E_k(p) = E_k^0(p)$ by induction for $1 \leq k \leq R$. Q.E.D.

5.12 *Definition:* The normal form $W(p)$ of $Z(p)$ is the matrix $\Psi_Z^{-1}(p)E(p)$. We write the elements of $W(p)$ in their lowest terms,

$$W(p) = A(p)Z(p)B(p) = \begin{bmatrix} \frac{e_1(p)}{\Psi_1(p)} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \frac{e_2(p)}{\Psi_2(p)} & \ddots & & & \\ \cdot & & & \frac{e_R(p)}{\Psi_R(p)} & & \\ \cdot & & & & & \\ \cdot & & & & 0 & \\ 0 & & & & & \ddots\, 0 \end{bmatrix} \tag{2}$$

with the polynomials $e_k(p)$, $\Psi_k(p)$ each having unit leading coefficients.

5.13 *Theorem:* The normal form $W(p)$ of $Z(p)$, as given by (2), has the properties (a′), (b′), (c′), (d′), and (e′) listed below. Furthermore, any $W^0(p)$, given by (2) with superscripts on W, A, B, e_k, and $\Psi_k(1 \leq k \leq R)$, which satisfies (a′), (b′), (c′), and (d′) with corresponding superscripts, is in fact $W(p)$.

(a′) R is the rank of $Z(p)$

(b′) For each k, $1 \leq k \leq R$, $e_k(p)$ and $\Psi_k(p)$ are relatively prime polynomials with unit leading coefficients.

(c′) Each $e_k(p)$ is a factor of $e_{k+1}(p)$, $1 \leq k \leq R - 1$, and each $\Psi_j(p)$ is a factor of $\Psi_{j-1}(p)$, $2 \leq j \leq R$.

(d′) $A(p)$ and $B(p)$ are polynomial matrices each with a constant non-vanishing determinant

(e′) $\Psi_1(p) = \Psi_Z(p)$.

Proof: (a′) and (d′) follow immediately from (a) and (d) of 5.1. (b′) is a matter of definition. (c′) follows from (c) of 5.1 and the definition, 5.12, since the effect of cancelling common factors in each fraction of the sequence

$$\frac{E_1(p)}{\Psi_Z(p)}, \frac{E_2(p)}{\Psi_Z(p)}, \cdots, \frac{E_R(p)}{\Psi_Z(p)}$$

cannot remove from any $E_k(p)$ a factor which was present in earlier $E_j(p)(j < k)$ but was not cancelled therefrom (treat each linear factor of Ψ_Z and of E_1 as distinct, and each linear factor of $\dfrac{E_{k+1}(p)}{E_k(p)}$ as distinct to see this easily).

Property (e′) is best proved by a reductio ad absurdum. We recall that $E_1(p)$ is the highest common factor of all elements of $\Psi_Z(p)Z(p)$. Suppose now that $E_1(p)$ contained a factor φ in common with $\Psi_Z(p)$. Then every non-zero element of $\Psi_Z(p)Z(p)$ contains the factor φ. Hence no denominator in $Z(p)$ cancels φ from $\Psi_Z(p)$. Hence no denominator contains φ as a factor, but this denies its presence in their least common multiple, $\Psi_Z(p)$.

The uniqueness of $W(p)$ follows at once from the uniqueness lemma, 5.11. Multiply (2) by $\Psi_Z(p)$. Then

$$\Psi_Z(p)W^0(p) = A^0(p)\Psi_Z(p)Z(p)B^0(p) \tag{3}$$

has diagonal elements of the form

$$\frac{\Psi_Z(p)e_k(p)}{\Psi_k(p)}, \qquad 1 \leq k \leq R. \tag{4}$$

But by (3) and (d′), these are the result of polynomial operations on the polynomial matrix $\Psi_Z(p)Z(p)$. Hence the elements (4) are polynomials, and each has unit leading coefficient. $\Psi_Z(p)W^0(p)$ then clearly satisfies (a), (b), (c), and (d) of 5.1. Therefore by 5.11, $\Psi_Z(p)W^0(p) = E(p) = \Psi_Z(p)W(p)$. Therefore $W^0(p) = W(p)$. Q.E.D.

5.14 *Corollary:* $W(p)$ is its own normal form.

5.15 *Corollary:* Let $\varphi(p)$ be a rational function and

$$Z_1(p) = \varphi(p)Z(p).$$

Let $W(p)$ be the normal form of $Z(p)$ and $W_1(p)$ the normal form of $Z_1(p)$. Then, when written in normalized lowest terms,

$$W_1(p) = \varphi(p)W(p).$$

Proof: Supposing that (2) above holds for W and Z, we have

$$\varphi(p)W(p) = A(p)Z_1(p)B(p).$$

Call the left side of this equation $W_1^0(p)$. We must identify this with $W_1(p)$. We have just showed that it satisfies (d′) of 5.13. It clearly satisfies (a′), (b′) and (c′), with Z_1 written for Z. Hence 5.13 implies the desired equality.

5.16 *Corollary:* If $C(p)$ and $D(p)$ are polynomial matrices with constant non-vanishing determinants, then the normal forms of $Z(p)$ and $C(p)Z(p)D(p)$ are the same.

Proof:

$$AZB = (AC^{-1})CZD(D^{-1}B)$$

and the bracketed factors are again polynomial matrices with constant non-vanishing determinants.

5.2 *Definition:* The point p_0 is a pole of $Z(p)$ if some element of $Z(p)$ has a pole at $p = p_0$. If p_0 is not a pole of $Z(p)$, we say that $Z(p_0)$ is finite, or that $Z(p)$ is finite at p_0.

5.21 If p_0 is a pole of $Z(p)$, we may expand each element of Z in partial fractions and collect those terms having poles at p_0, obtaining, when $p_0 \neq \infty$,

$$Z(p) = (p - p_0)^{-r}Z_r + (p - p_0)^{-r+1}Z_{r-1} + \cdots + (p - p_0)^{-1}Z_1 + Z_0(p), \tag{1}$$

where $Z_0(p_0)$ is finite, $Z_r \neq 0$, and the Z_k, $1 \leq k \leq r$, are matrices of constants. If $p_0 = \infty$, we read p^{ℓ} for $(p - p_0)^{-\ell}$ in (1), $1 \leq \ell \leq r$. All of $Z_0(p)$, $Z_1, \cdots, Z_r$ are uniquely defined by their constructionfrom $Z(p)$.

5.22 *Definition:* If $Z(p)$ is given by (1) above, then r is the order of the pole of $Z(p)$ at p_0.

5.23 Clearly, if $Z(p)$ has the form (1) at $p_0 \neq \infty$, some non-vanishing element of $Z(p)$ has a denominator containing the factor $(p - p_0)^r$, and no element has a pole of order higher than r at p_0. Hence $(p - p_0)^r$ divides $\Psi_Z(p)$, but no higher power of $(p - p_0)$ does. Therefore, by (c′) of 5.13, the normal form $W(p)$ of $Z(p)$ has a first element with an r^{th} order pole at p_0. In particular, then, $p_0 \neq \infty$ is a pole of order r of $Z(p)$ if and only if it is a pole of order r of $W(p)$.

5.24 *Definition:* Consider a pole of order r of $Z(p)$, say p_0, with $p_0 \neq \infty$. In the normal form $W(p)$ of $Z(p)$, (2) of 5.12, let γ_k be the order of the pole of the k^{th} diagonal element

$$\frac{e_k(p)}{\Psi_k(p)}$$

at the point $p = p_0$. Then $\gamma_k \geq \gamma_{k+1}$, and $\gamma_1 = r$. We write the γ_k in an ordered array

$$S(Z, p_0) = [\gamma_1, \gamma_2, \cdots, \gamma_n].$$

5.25 *Definition:* Consider two matrices $Z(p)$ and $Z_1(p)$, with

$$S(Z, p_0) = [\gamma_1, \gamma_2, \cdots, \gamma_n],$$
$$S(Z_1, p_0) = [\gamma_1', \gamma_2', \cdots, \gamma_n'].$$

We say

$$S(Z, p_0) \geq S(Z_1, p_0) \tag{2}$$

if and only if

$$\gamma_1 + \gamma_2 + \cdots + \gamma_k \geq \gamma_1' + \gamma_2' + \cdots + \gamma_k'$$

for every $k = 1, 2, \cdots, n$. We say

$$S(Z, p_0) = S(Z_1, p_0) \tag{3}$$

if

$$\gamma_k = \gamma_k'$$

for $k = 1, 2, \cdots, n$. It is easy to see that (3) is equivalent to the simultaneous validity of (2) and the reverse inequality.

5.26 *Theorem:* Let $p_0 \neq \infty$ be a pole of $Z(p)$. Let $F(p)$ be a rational $n \times n$ matrix which is finite at p_0. Then

$$S(Z, p_0) \geq S(FZ, p_0).$$

In particular, if $F(p)$ is also non-singular at p_0, then

$$S(Z, p_0) = S(FZ, p_0).$$

Proof: Let $\psi_F(p)$ and $\psi_Z(p)$ be the least common denominators of the elements of $F(p)$ and $Z(p)$, respectively. Then the exponent of $(p - p_0)$ in $\psi_Z(p)$ is r, while in $\psi_F(p)$ it is zero by 5.23.

Let $-\varepsilon_k$ be the exponent of $(p - p_0)$ in the k^{th} diagonal element of the normal form of Z, and $-\varepsilon_k'$ the similar quantity for FZ. Then

$$\begin{aligned} \varepsilon_1 \geq \varepsilon_2 \geq \cdots \geq \varepsilon_n, \\ \varepsilon_1' \geq \varepsilon_2' \geq \cdots \geq \varepsilon_n', \end{aligned} \tag{3}$$

by (e′) of 5.13. Let

$$\gamma_k = \sup(\varepsilon_k, 0),$$
$$\gamma_k' = \sup(\varepsilon_k', 0),$$

Then $\gamma_k \geq \varepsilon_k$, $\gamma_k' \geq \varepsilon_k'$, and

$$S(Z, p_0) = [\gamma_1, \gamma_2, \cdots, \gamma_n],$$
$$S(FZ, p_0) = [\gamma_1', \gamma_2', \cdots, \gamma_n'].$$

By 5.15, the normal form of FZ is

$$(\psi_F\psi_Z)^{-1} \cdot (\text{normal form of } \psi_F\psi_Z FZ).$$

Hence the exponent of $(p - p_0)$ in the k^{th} diagonal element of the normal form of $\psi_F\psi_Z FZ$ is $r - \varepsilon_k'$. By a similar argument, the exponent of $(p - p_0)$ in the k^{th} diagonal element of the normal form of $\psi_Z Z$ is $r - \varepsilon_k$. Hence, by (e) of 5.1,

$$(r - \varepsilon_1') + \cdots + (r - \varepsilon_b')$$

is the exponent of $(p - p_0)$ in the highest common factor of all b-rowed minor determinants of $\psi_F\psi_Z FZ$. Similarly

$$(r - \varepsilon_1) + \cdots + (r - \varepsilon_b)$$

is the exponent of $(p - p_0)$ in the highest common factor of all b-rowed minor determinants of $\psi_Z Z$.

Now $\psi_F\psi_Z FZ$ is a polynomial matrix. A typical b-rowed minor determinant of this matrix is of the form

$$\psi_F^b \psi_Z^b \sum M_b N_b, \tag{4}$$

where the summation is over certain products $M_b N_b$ of b-rowed minors M_b of F and b-rowed minors N_b of Z. For a proof of this, see MacDuffee[16], Theorem 99.1. The expression (4) is the same as

$$\sum (\psi_F^b M_b)(\psi_Z^b N_b) \tag{5}$$

where the factors $(\psi_Z^b N_b)$ are now b-rowed minors of $\psi_Z Z$. If φ is a factor common to all b-rowed minors of $\psi_Z Z$, it certainly is a factor common to all expressions (4) or (5). Hence the highest common factor of all b-rowed minor determinants of $\psi_F\psi_Z FZ$—i.e., of all expressions (4) or (5),—has an exponent for $(p - p_0)$ no lower than that in the highest common factor of all b-rowed minor determinants of $\psi_Z Z$. Hence for any b,

$$(r - \varepsilon_1') + \cdots + (r - \varepsilon_b') \geq (r - \varepsilon_1) + \cdots + (r - \varepsilon_b),$$

or

$$\varepsilon_1 + \cdots + \varepsilon_b \geq \varepsilon_1' + \cdots + \varepsilon_b'.$$

It follows that

$$\gamma_1 + \cdots + \gamma_b \geq \varepsilon_1' + \cdots + \varepsilon_b'.$$

This being true for every b, it is certainly true for every b such that all terms on the right are ≥ 0 (cf. (2)). This means that for $b = 1$, and for

every successive $b > 1$ such that $\varepsilon_b' \geq 0$,

$$\gamma_1 + \cdots + \gamma_b \geq \gamma_1' + \cdots + \gamma_b' .$$

This inequality is now not altered if non-negative numbers are added to its left member and zeros to its right member. Hence it holds for all b, $1 \leq b \leq n$, and

$$S(Z, p_0) \geq S(FZ, p_0). \tag{6}$$

This is the first claim of the theorem.

Now if $F(p)$ is non-singular at p_0, then $F^{-1}(p)$ is rational, and finite at p_0. Hence by what is already proved,

$$S(FZ, p_0) \geq S(F^{-1}(FZ), p_0).$$

This last array is just $S(Z, p_0)$. Hence we have (6) and its reverse, and the theorem is proved.

5.27 *Theorem:* If $p_0 \neq \infty$ and

$$Z(p) = Z_1(p) + Z_2(p),$$

where $Z_2(p)$ is finite at p_0, then

$$S(Z, p_0) = S(Z_1, p_0).$$

The proof of this depends upon the following lemma.

5.28 *Lemma:* Let $Z^*(p)$ be such that at $p = p_0 \neq \infty$ its only elements having poles are on the main diagonal. Let $-\varepsilon_1', -\varepsilon_2', \cdots$ be the exponents of $(p - p_0)$ in the diagonal elements of $Z^*(p)$, so enumerated that

$$+\varepsilon_1' \geq +\varepsilon_2' \geq \cdots \geq +\varepsilon_n' .$$

Let $-\varepsilon_1, -\varepsilon_2, \cdots, -\varepsilon_n$ be the exponents of $(p - p_0)$ in the successive diagonal elements of the normal form of $Z^*(p)$. Then if $\varepsilon_b' \geq 0$ we have

$$\varepsilon_1' + \cdots + \varepsilon_b' \geq \varepsilon_1 + \cdots + \varepsilon_b .$$

Proof: There exist constant non-singular matrices F, G such that FZ^*G has the same rows and columns as Z^* so permuted that the diagonal elements of FZ^*G are arranged in the order of ascending powers of $(p - p_0)$, the highest order pole being in the first position. Since the normal forms of Z^* and FZ^*G are identical, it suffices to consider Z^* itself to be in this form.

Let $\psi = \psi_{Z^*}(p)$. Now ψZ^* has its diagonal elements in the order of increasing positive power of $(p - p_0)$. Furthermore, any off-diagonal element of ψZ^* has $(p - p_0)^r$ as a factor.

Let b be such that $\varepsilon_b' \geq 0$. Any b-rowed minor of ψZ^* is a sum of products of b elements of ψZ^*. That b-rowed minor which has in it a term with a lowest possible exponent of $(p - p_0)$ is the upper left b-rowed minor. Even this minor has a term with exponent

$$(r - \varepsilon_1') + \cdots + (r - \varepsilon_b') \tag{7}$$

for $(p - p_0)$, this term being the product of the main diagonal elements. Hence the highest common factor of all b-rowed minors of ψZ^* has an exponent for $(p - p_0)$ not less than (7). Hence

$$(r - \varepsilon_1) + \cdots + (r - \varepsilon_b) \tag{8}$$

is not less than (7), since this is the exponent of $(p - p_0)$ in the product of the first b diagonal elements of the normal form of ψZ^*. The inequality between (8) and (7) is just the conclusion claimed in the lemma.

5.281 *Proof* of 5.27: Let

$$W(p) = A(p)Z(p)B(p)$$

be the normal form of $Z(p)$. Then

$$W = AZ_1B + AZ_2B. \tag{9}$$

If we expand all three terms here in Laurent series about p_0, the term AZ_2B contributes no negative powers. It follows then from the diagonal form of W that the matrix

$$Z^* = AZ_1B$$

satisfies the conditions of 5.28. The ε_k' of that lemma are, from (9), just the exponents of $(p - p_0)$ in the successive diagonal elements of W, the normal form of Z, and the ε_k of 5.28 are the similar quantities for the normal form of $Z^* = AZ_1B$. But the normal form of AZ_1B is the same as that of Z_1 (5.16). Therefore in the inequality of 5.28 we may interpret all of the ε's as exponents in the respective normal forms of Z and Z_1.

Now

$$Z_1(p) = Z(p) + (-Z_2(p))$$

and $-Z_2(p)$ is again finite at p_0. Hence we may conclude by the argument just used that if $\varepsilon_b \geq 0$ also

$$\varepsilon_1 + \cdots + \varepsilon_b \geq \varepsilon_1' + \cdots + \varepsilon_b'.$$

Hence if either of ε_b or ε_b' is non-negative

$$\varepsilon_1 + \cdots + \varepsilon_b = \varepsilon_1' + \cdots + \varepsilon_b'.$$

By induction on b, then,

$$\varepsilon_k = \varepsilon_k'$$

for $k = 1, 2$, etc. until such k that both are negative. Therefore

$$\gamma_k = \sup(\varepsilon_k, 0) = \gamma_k' = \sup(\varepsilon_k', 0)$$

for all $k = 1, 2, \cdots, n$. That is,

$$S(Z, p_0) = S(Z_1, p_0),$$

Q.E.D.

5.29 *Theorem:* Let $Z(p)$ be such that at $p = p_0 \neq \infty$ its only elements having poles lie on the main diagonal. Let $\sigma_1, \sigma_2, \cdots, \sigma_n$ be the orders of these poles, so enumerated that

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n .$$

Then

$$S(Z, p_0) = [\sigma_1, \sigma_2, \cdots, \sigma_n].$$

Proof: We write

$$Z(p) = Z^*(p) + Z_2(p),$$

where $Z^*(p)$ is diagonal, having exactly the diagonal elements of $Z(p)$. By 5.27,

$$S(Z, p_0) = S(Z^*, p_0).$$

Now $Z^*(p)$ falls under 5.28, but is diagonal in addition. In the proof of 5.28, therefore, it is exactly the principal minors of ψZ^* which have the lowest exponents for $(p - p_0)$, since all non-principal minors vanish and have zeros of arbitrary order at $p = p_0$. Furthermore, (7) is exactly the least exponent of $(p - p_0)$ in any b-rowed minor of ψZ^* since the principal minors are simple products. Hence (7) and (8) are equal, for any $b = 1, 2, \cdots, n$. Therefore the exponents in the normal form of Z^* are exactly those of Z^* and

$$S(Z, p_0) = S(Z^*, p_0) = [\sigma_1, \sigma_2, \cdots, \sigma_n].$$

Q.E.D.

5.3. *Definition:* Let

$$p = T(q) = \frac{\alpha q + \beta}{\gamma q + \delta}$$

be a non-singular bi-rational transformation from the q-sphere to the

p-sphere. Denote its inverse by

$$q = T^{-1}(p).$$

Given a rational $Z(p)$ the matrix

$$Z_1(q) = Z(T(q))$$

is rational in q.

For any p_0 such that $T^{-1}(p_0) \neq \infty$, we define

$$S_T(Z, p_0) = S(Z_1, T^{-1}(p_0)).$$

5.31 *Theorem:* If p_0 and $T^{-1}(p_0)$ are both finite,

$$S_T(Z, p_0) = S(Z, p_0).$$

Proof: Let $W_1(q)$ be the normal form of

$$Z_1(q) = Z(T(q)).$$

We have

$$W_1(q) = A(q)Z_1(q)B(q).$$

Consider

$$W_2(p) = W_1(T^{-1}(p)) = A(T^{-1}(p))Z(p)B(T^{-1}(p)).$$

Here the pre- and post factors of $Z(p)$ are rational, finite, and non-singular at p_0. Hence by 5.26

$$S(W_2, p_0) = S(Z, p_0). \tag{1}$$

Let $q_0 = T^{-1}(p_0)$. It is then easily computed that the inverse transformation $T^{-1}(p)$ takes tbe form

$$q - q_0 = \frac{a(p - p_0)}{b(p - p_0) + 1}, \qquad a \neq 0.$$

Any given diagonal element of $W_1(q)$ is of the form

$$(q - q_0)^{\epsilon} R(q),$$

where ε may have any sign, and $R(q)$ is rational, finite, and not zero at q_0. The corresponding diagonal element of $W_2(p)$ is then

$$(p - p_0)^{\epsilon} \left(\frac{a}{b(p - p_0) + 1}\right)^{\epsilon} R_1(p),$$

where $R_1(p) = R(T^{-1}(p))$, and the factor multiplying $(p - p_0)^{\epsilon}$ is again

finite and not zero at p_0. The exponents of $(p - p_0)$ in the elements of $W_2(p)$ are therefore exactly the exponents of $q - q_0$ in the elements of $W_1(q)$. From 5.29, then

$$S(W_2, p_0) = S(W_1, q_0).$$

This with (1) and the definition 5.3 proves the theorem.

5.32 *Definition:* Given any p_0, let $p = T(q)$ be a non-singular bi-rational transformation such that $q_0 = T^{-1}(p_0) \neq \infty$. We define $S^*(Z, p_0)$ by

$$S^*(Z, p_0) = S_T(Z, p_0).$$

5.33 *Lemma:* $S^*(Z, p_0)$ is independent of the T chosen to define it.

Proof: Consider $q = T^{-1}(p)$ and $r = U^{-1}(p)$, each such that p_0 is mapped on a finite point. Then by definition

$$S_T(Z, p_0) = S(Z_1, q_0),$$

$$S_U(Z, p_0) = S(Z_2, r_0),$$

where

$$q_0 = T^{-1}(p_0), \qquad r_0 = U^{-1}(p_0),$$

$$Z_1(q) = Z(T(q)),$$

$$Z_2(r) = Z(U(r)).$$

Now $r = U^{-1}(T(q)) = V(q)$, say, and r_0 and q_0 are finite. Hence by 5.31

$$S_V(Z_2, r_0) = S(Z_2, r_0) = S_U(Z, p_0). \tag{2}$$

But by definition

$$S_V(Z_2, r_0) = S(Z_3, V^{-1}(r_0)) = S(Z_3, q_0) \tag{3}$$

where

$$Z_3(q) = Z_2(V(q))$$

But

$$Z_2(V(q)) = Z(U(U^{-1}(T(q)))) = Z(T(q)) = Z_1(q).$$

Hence

$$S(Z_3, q_0) = S(Z_1, q_0) = S_T(Z, p_0).$$

This, with (2) and (3), proves the lemma.

5.34 *Theorem:* Theorems 5.26, 5.27, and 5.29 hold for S^* without the restriction that p_0 be finite.

Proof: Let $q_0 = T^{-1}(p_0) \neq \infty$. For 5.26 we have

$$S^*(Z, p_0) = S(Z_1, q_0) \geq S(F_1 Z_1, q_0) = S^*(FZ, p_0)$$

where the equalities are by definition and the inequality is 5.26 applied to matrices rational in q, since

$$F_1(q) = F(T(q))$$

is by hypothesis finite at q_0. The remaining conclusion of 5.26 follows similarly. The proofs of 5.27 and 5.29 are equally simple.

5.35 *Theorem:* If we extend 5.3 to S^* by defining

$$S_T^*(Z, p_0) = S^*(Z_1, T^{-1}(p_0)),$$

then 5.31 holds for S^* with no restrictions on p_0 or $T^{-1}(p_0)$.

Proof: By their definitions,

$$S_T^*(Z, p_0) = S^*(Z_1, T^{-1}(p_0)) = S_U(Z_1, T^{-1}(p_0)), \tag{4}$$

where U is such that $U^{-1}(T^{-1}(p_0))$ is finite. But

$$S_U(Z_1, T^{-1}(p_0)) = S(Z_2, U^{-1}(T^{-1}(p_0))) \tag{5}$$

where

$$Z_2(r) = Z_1(U(r)) = Z(T(U(r))).$$

Let $V(r) = T(U(r))$. Then, by definitions,

$$S(Z_2, U^{-1}(T^{-1}(p_0))) = S_V(Z, p_0) = S^*(Z, p_0), \tag{6}$$

since $V^{-1}(p_0) = U^{-1}(T^{-1}(p_0))$ is finite. The theorem follows from (4), (5), and (6).

5.4 *Definition:* Let

$$S^*(Z, p_0) = [\gamma_1, \gamma_2, \cdots, \gamma_n].$$

Define

$$\delta(Z, p_0) = \gamma_1 + \gamma_2 + \cdots + \gamma_n,$$

$$\delta(Z) = \sum \delta(Z, p_0),$$

where the latter summation is over all poles p_0 of $Z(p)$, including $p_0 = \infty$. This $\delta(Z)$ is the degree of Z for which we must establish the properties claimed in 2.11 through 2.17. These properties will be demonstrated in 5.41 through 5.45, in numerical order, saving 2.13, which is deferred to 5.46.

5.41 Clearly $\delta(Z)$ is an integer and non-negative. If $\delta(Z) = 0$, then every γ at every p_0 is zero. Hence no p_0, not even ∞, is a pole of Z. Hence each element of $Z(p)$ is a constant. This estahlishes 2.11 and 2.12.

5.42 Suppose

$$Z(p) = Z_1(p) + Z_2(p)$$

where each $Z_i(p)$ is finite at every pole of the other. The poles of $Z(p)$ are then exactly the poles $p_0^{(1)}$ of Z_1 and those $p_0^{(2)}$ of Z_2. At each pole, 5.27 applies in the enlarged sense of 5.34, so

$$\delta(Z, p_0^{(i)}) = \delta(Z_i, p_0^{(i)}).$$

Breaking the sum defining $\delta(Z)$ into sums over the $p_0^{(1)}$ and $p_0^{(2)}$ proves that

$$\delta(Z) = \delta(Z_1) + \delta(Z_2).$$

This is 2.14.

5.43 If

$$Z(p) = f(p)R,$$

where R is a constant matrix, then the normal form of $Z(p)$ is $f(p)$ times a diagonal matrix of the same rank as R (5.15). 2.15 then follows at once.

5.44 If

$$Z_1(p) = AZ(p)B,$$

where A and B are constant and non-singular, the poles of $Z_1(p)$ and $Z(p)$ are the same. At each, 5.26 applies in the enlarged sense of 5.34. Therefore $\delta(Z_1) = \delta(Z)$. This is 2.16.

5.45 If $Z_1(p)$ is $Z(p)$ bordered by zeros, they have the same poles. One verifies at onee from 5.11 that the normal form of $Z_1(p)$ is that of $Z(p)$ bordered by zeros. Since also $Z_1(T(q))$ is $Z(T(q))$ bordered hy zeros, it follows that

$$S^*(Z_1, p_0) = S^*(Z, p_0)$$

at every pole, whence $\delta(Z_1) = \delta(Z)$. This is 2.17.

5.46 We must prove that if $Z(p)$ is non-singular, then

$$\delta(Z) = \delta(Z^{-1})$$

Proof: Choose a bi-rational transformation $p = T(q)$ sueh that at

$p = T(\infty)$ both of $Z(p)$ and $Z^{-1}(p)$ are finite. Let

$$Z_1(q) = Z(T(q)).$$

Then

$$Z_1^{-1}(q) = Z^{-1}(T(q)).$$

Let $W_1(q)$ be the normal form of $Z_1(q)$, with diagonal elements

$$\frac{e_k(q)}{\psi_k(q)}$$

in lowest terms. Since $Z_1(q)$ is of rank n, none of these vanish identically.

We first claim that $\delta(Z) = \delta(Z_1)$. The poles p_0 of Z are exactly the points

$$p_0 = T(q_0)$$

where q_0 runs over the poles of Z_1. At each pole,

$$S^*(Z, p_0) = S_T^*(Z, p_0) = S^*(Z_1, q_0)$$

by 5.35. Hence $\delta(Z, p_0) = \delta(Z_1, q_0)$ and the result follows by addition. Similarly, then, $\delta(Z^{-1}) = \delta(Z_1^{-1})$.

Next we assert that $\delta(Z_1)$ is just the degree of the polynomial

$$\psi_1(q)\psi_2(q) \cdots \psi_n(q).$$

For $\delta(Z_1, q_0)$ is the exponent of $(q - q_0)$ in this polynomial, and the zeros of this polynomial are exactly the poles of $Z_1(q)$.

We observe that if

$$W_1(q) = A(q)Z_1(q)B(q),$$

then

$$W_1^{-1}(q) = B^{-1}(q)Z_1^{-1}(q)A^{-1}(q).$$

This then is the result of polynomial operations on $Z_1^{-1}(q)$, and has diagonal elements

$$\frac{\psi_k(q)}{e_k(q)}. \tag{1}$$

Clearly by arranging these in reverse order, we have a normal form. This is 5.13. Hence the functions (1) are the diagonal elements of the normal form of $Z_1^{-1}(q)$. The argument above applied to $Z_1^{-1}(q)$ then shows that $\delta(Z_1^{-1})$ is the degree of

$$e_1(q) \cdots e_n(q).$$

Finally, we note the determinant relation

$$| W_1(q) | = | A(q) || Z_1(q) || B(q) | = (\text{constant}) \times | Z_1(q) |,$$

since the determinants of A and B are constant. Now $Z_1(q)$ has no pole at $q = \infty$, hence its determinant is finite there. The same is true of $Z_1^{-1}(q)$, so indeed

$$| Z_1(\infty) | \neq 0.$$

Now by direct calculation

$$| W_1(q) | = \frac{e_1(q) \cdots e_n(q)}{\psi_1(q) \cdots \psi_n(q)}.$$

Since this is finite and not zero at $q = \infty$, the numerator and denominator are of the same degree. Hence

$$\delta(Z) = \delta(Z_1) = \text{degree } (\Pi\psi_k) = \text{degree } (\Pi e_k) = \delta(Z_1^{-1}) = \delta(Z^{-1}).$$

VI. THE EXACT COUNT OF REACTIVE ELEMENTS

6.0 We showed in the inductive argument of 4.07 that the Brune process constructs a realization for a given $Z(p)$ which uses exactly $\delta(Z)$ reactive elements. To establish 2.18, we must still show that no network with fewer than $\delta(Z)$ reactive elements can do this. To prove this, we shall show that if $Z(p)$ is the impedance matrix of a network containing x reactive elements, then

$$\delta(Z) \leq x. \tag{1}$$

We shall, in fact, in this Section show somewhat more than (1). The demonstration of (1) requires enough calculation that is as easy to prove the following extension of 2.18.

6.01 *Theorem:* Given any linear correspondence L, (I, 6.2), which PR, (I, 16.71), there exists a number $\delta(L)$ such that

(i) The realization process outlined in (I, 8) and 4.07 of this Part constructs with $\delta(L)$ reactive elements a network realizing a member of the Cauer class of L.

(ii) If L is the correspondence established by the Cauer class of a physical network which contains x reactive elements, then

$$\delta(L) \leq x.$$

The proof is divided among the remaining paragraphs of this Section. We maintain here a strict distinction between geometric objects and their concrete coordinate representations.

6.02 We observe at once that if a $\delta(L)$ exists satisfying (i) and (ii), then it must be unique because it is exactly the minimum number of reactive elements required to realize any representative of the Cauer class L. No particular pains then need be taken as we go along to verify that the value of $\delta(L)$ arrived at is in fact independent of the mode of defining it.

6.1 Given a PR geometrical linear correspondence L between $\mathbf{V}$ and $\mathbf{K}$, there is a frame which reduces L in the sense of (I, 13.02). In this frame we have the dual decomposition

$$\mathbf{V} = \mathbf{V}_{L0} \oplus \mathbf{V}_2 \oplus \mathbf{V}_1$$

$$\mathbf{K} = \mathbf{K}_1 \oplus \mathbf{K}_2 \oplus \mathbf{K}_{L0}$$

in which each subspace is real and spanned by selected basis vectors. Furthermore,

$$\mathbf{V}_L = \mathbf{V}_{L0} \oplus \mathbf{V}_2 ,$$

$$\mathbf{K}_L = \mathbf{K}_2 \oplus \mathbf{K}_{L0} ,$$

Finally, if r is the dimension of $\mathbf{V}_2$ and $\mathbf{K}_2$, there is an $r \times r$ PR matrix $[Z_1(p)]$ such that, when

$$[v_2 , k_2] \in L(p)$$

and

$$v_2 \in \mathbf{V}_2 , \qquad k_2 \in \mathbf{K}_2 ,$$

then

$$[v_2] = [Z_1(p)][k_2].$$

Here the r-tuples are those representing v_2 and k_2 as elements of $\mathbf{V}_2$ and $\mathbf{K}_2$ in the chosen frame.

6.11 *Definition:* We define $\delta(L)$ by

$$\delta(L) = \delta([Z_1]),$$

where $[Z_1(p)]$ is the matrix described above.

6.12 This number $\delta(L)$ is the number of reactive elements used when the Brune process is applied to realize $[Z_1(p)]$. (This is 4.07). Then, however, by the argument of (I, 8.5), the representative $[L]$ of L in the particular frame in question can be realized by adjoining open and short circuits to a realization of $[Z_1(p)]$. This operation adds no new reactive elements. Neither does the operation of converting $[L]$ to any

Cauer equivalent $[L]_1$ by the use of ideal transformers. Therefore the particular $\delta(L)$ we have defined—which depends for its definition upon a somewhat arbitrary choice of coordinate frame—satisfies (i) of 6.01.

6.2 *Lemma:* Let L be a PR geometrical linear correspondence between $\mathbf{K}$ and $\mathbf{V}$, and M another between spaces $\mathbf{J}$ and $\mathbf{U} = \mathbf{J}^*$ obtained by restricting L as in (I, 18). Then

$$\delta(M) \leq \delta(L).$$

Proof: We use the results and notation of (I, 18). In particular, C is a real constant operator from $\mathbf{J}$ to $\mathbf{K}$, C^* its adjoint from $\mathbf{V}$ to $\mathbf{U}$, and the pairs of $M(p)$ are those pairs

$$[u, j]$$

such that

$$u = C^*v \qquad \text{and} \qquad [v, Cj] \in L(p).$$

Choose a frame in $\mathbf{V}$ and $\mathbf{K}$ which reduces L as in 6.1. We recall that $\mathbf{J}_M$ consists of all vectors $j \in \mathbf{J}$ such that $Cj \in \mathbf{K}_L$ (I, 18.31). Let $\mathbf{J}_2$ consist of all $j \in \mathbf{J}$ such that

$$Cj \in \mathbf{K}_2 .$$

Let $\mathbf{J}_3$ consist of all $j \in \mathbf{J}$ such that

$$Cj \in \mathbf{K}_{L0} .$$

Then $\mathbf{J}_2$ and $\mathbf{J}_3$ are disjoint and both are subspaces of $\mathbf{J}_M$. We can therefore write

$$\mathbf{J}_M = \mathbf{J}_2 \oplus \mathbf{J}_3 \oplus \mathbf{J}_4 ,$$

after a suitable choice of $\mathbf{J}_4$.

We now claim that

$$\mathbf{J}_3 \oplus \mathbf{J}_4 \subset \mathbf{J}_{M0} . \tag{1}$$

For we have if $j \in \mathbf{J}_M$ that, uniquely,

$$j = j_2 + j_3 + j_4 ,$$

where $j_i \in \mathbf{J}_i$. Therefore

$$Cj = Cj_2 + Cj_3 + Cj_4$$

where by construction $Cj_2 \in \mathbf{K}_2$, $Cj_3 \in \mathbf{K}_{L0}$, and, necessarily, then $Cj_4 = 0$. If $j_2 = 0$, therefore, $Cj \in \mathbf{K}_{L0}$ and

$$[0, Cj] \in L(p).$$

Therefore

$$[C^*0, j] = [0, j] \in M(p).$$

this proves (1).

We can now write

$$\mathbf{J}_M = \mathbf{J}_{21} \oplus \mathbf{J}_{20} \oplus \mathbf{J}_0 \tag{2}$$

where

$$\begin{aligned} \mathbf{J}_2 &= \mathbf{J}_{21} \oplus \mathbf{J}_{20}, \\ \mathbf{J}_{20} &= \mathbf{J}_2 \cap \mathbf{J}_{M0}, \\ \mathbf{J}_0 &= \mathbf{J}_3 \oplus \mathbf{J}_4, \\ \mathbf{J}_{M0} &= \mathbf{J}_{20} \oplus \mathbf{J}_0 . \end{aligned} \tag{3}$$

Choosing an arbitrary $\mathbf{J}_5$ disjoint from $\mathbf{J}_M$, we can write, using (2) and (3),

$$\mathbf{J} = \mathbf{J}_5 \oplus \mathbf{J}_{21} \oplus \mathbf{J}_{M0}, \tag{4}$$

where

$$\mathbf{J}_M = \mathbf{J}_{21} \oplus \mathbf{J}_{M0} .$$

Using the arguments of (I, 12.3), we find that the decomposition of $\mathbf{U}$ dual to (4) is, because M is PR,

$$\mathbf{U} = \mathbf{U}_{M0} \oplus \mathbf{U}_{21} \oplus \mathbf{U}_3 \tag{5}$$

where

$$\mathbf{U}_M = \mathbf{U}_{M0} \oplus \mathbf{U}_{21} .$$

As in (I, 12.3) we can now introduce a frame appropriate to the decomposition indicated in (4) and (5) and obtain a matrix $[Z_{21}(p)]$ describing the correspondence between $\mathbf{J}_{21}$ and $\mathbf{U}_{21}$. Say this is an $m \times m$ matrix, m being the dimension of $\mathbf{J}_{21}$. We can define

$$\delta(M) = \delta([Z_{21}]).$$

Let $\mathbf{J}_2$ have dimension m_1. By (3), if we border $[Z_{21}(p)]$ by $m_1 - m$ rows and columns of zeros, to obtain an $m_1 \times m_1$ matrix $[Z_2(p)]$, we can interpret $[Z_2(p)]$ as follows:

Given $j \in \mathbf{J}_2$, it can be represented by an m_1-tuple $[j]$ in the basis in that subspace. Then the m_1-tuple

$$[u] = [Z_2(p)][j] \tag{6}$$

represents in the dual basis in $(\mathbf{J}_2)^0$ a vector $u \in \mathbf{U}_{21}$ such that

$$[u, j] \in M(p).$$

Now this u necessarily is of the form

$$u = C^* v, \tag{7}$$

where

$$[v, Cj] \in L(p).$$

But $j \in \mathbf{J}_2$, so $Cj \in \mathbf{K}_2$, so v may be taken to be an element of $\mathbf{V}_2$, with components

$$[v] = [Z_1(p)][Cj] \tag{8}$$

in the basis therein.

We have bases now in $\mathbf{V}$, $\mathbf{K}$, $\mathbf{U}$, and $\mathbf{J}$, each of which has a set of basis vectors spanning, respectively, $\mathbf{V}_2$, $\mathbf{K}_2$, $(\mathbf{J}_2)^0$, and $\mathbf{J}_2$. By definition of $\mathbf{J}_2$, and by (7) and (8), C operates from $\mathbf{J}_2$ to $\mathbf{K}_2$, and C^* from $\mathbf{V}_2$ to $(\mathbf{J}_2)^0$. Hence in these respective bases C and C^* may be represented by $m_1 \times m_1$ matrices. In these bases then, from (7) and (8),

$$[u] = [C^*][v] = [C^*][Z_1(p)][C][j].$$

Comparing this with (6), we have

$$[Z_2(p)] = [C^*][Z_1(p)][C].$$

Hence by definitions and 5.26,

$$\delta(M) = \delta([Z_2]) \leq \delta([Z_1)]) = \delta(L).$$

This is the assertion to be proved.

6.3 We can now turn to (ii) of 6.01. We follow the synthesis procedure of (I, 19), as modified in the remarks of 3.2.

Consider a network constructed from x reactive elements, r resistors, and some ideal transformers. As in (I, 19.2), the synthesis of this network begins by juxtaposing the $r + x$ two poles and the ideal transformers, all as separate devices. The correspondence $[L]$ established by this juxtaposition is exhibited in (I, 19.2) as one described by a diagonal matrix $[Z_d(p)]$ juxtaposed with one described by certain ideal transformers. A frame which reduces this correspondence as in 6.1 can be found by a change of basis wholly within those subspaces in which the ideal transformers operate. Hence the degree $\delta(L)$ of this correspondence in exactly $\delta([Z_d])$ which, by 5.29, is x.

Now let $[M]$ be the concrete linear correspondence established by the

network to be synthesized. Then (I, 19.3, 19.4) show that $[M]$ is obtained by two successive restrictions upon $[L]$. Hence by 6.2

$$\delta(M) \leq \delta(L) = x.$$

Q.E.D.

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